THE FINE STRUCTURE OF TRANSLATION FUNCTORS

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Abstract. Let E be a simple finite dimensional representation of a semisimple Lie algebra with extremal weight ν and let $0 \neq e \in E_\nu$. Let $M(\tau)$ be the Verma module with highest weight τ and $0 \neq v_\tau \in M(\tau)_\tau$. We investigate the projection of $e \otimes v_\tau \in E \otimes M(\tau)$ on the central character $\chi(\tau+\nu)$. This is a rational function in τ and we calculate its poles and zeros. We then apply this result in order to compare translation functors.

Contents

1. Introduction

Let k be a field of characteristic zero and let $\mathfrak{g}$ be a split semisimple Lie algebra over k. Then $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{g}$ denote a Cartan and a Borel subalgebra, $\mathfrak{U} = \mathfrak{U}(\mathfrak{g})$ the universal enveloping algebra and $\mathfrak{Z} \subset \mathfrak{U}$ its center. We set $R^+ \subset R \subset \mathfrak{h}^*$ to be the roots of $\mathfrak{b}$ and of $\mathfrak{g}$. Let $\mathfrak{n}$ (resp. $\mathfrak{n}^-$) be the sum of the positive (negative) weight spaces and denote by $P^+ \subset P \subset \mathfrak{h}^*$ the set

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of dominant and the lattice of integral weights respectively. Let $\mathcal{W}$ be the Weyl group.
A $\mathfrak{g}$-module M is called $\mathfrak{Z}$-*finite*, if $\dim_k(\mathfrak{Z}m) < \infty$ for all $m \in M$. Every $\mathfrak{Z}$-finite module M splits under the operation of the center $\mathfrak{Z}$ into a direct sum of submodules $M = \oplus_{\chi \in \mathrm{Max}\mathfrak{Z}} M_\chi$. Here χ runs over the maximal ideals in the center and $M_\chi \in \mathcal{M}^\infty(\chi)$, where

$$\mathcal{M}^\infty(\chi) := \{M \mid \text{ for all } m \in M \text{ there exists } n \in \mathbb{N} \text{ such that } \chi^n m = 0\}.$$

We denote the projection onto the central character χ by $\mathrm{pr}_\chi : M \mapsto M_\chi$. By means of the Harish-Chandra homomorphism [Di, 7.4] $\xi : \mathfrak{Z} \to S(\mathfrak{h})$ (normalized by $z - \xi(z) \in \mathfrak{U}\mathfrak{n}\ \forall z \in \mathfrak{Z}$) we assign to each weight $\tau \in \mathfrak{h}^*$ its central character $\chi(\tau)$ defined by $\chi_\tau(z) := \big(\xi(z)\big)(\tau)\ \forall z \in \mathfrak{Z}$.
Let now $\nu \in P$ be an integral weight and denote by $E := E(\nu)$ the irreducible finite dimensional $\mathfrak{g}$-module with highest weight in $\mathcal{W}\nu$. It is known that for each $\mathfrak{Z}$-finite module M the tensor product $E \otimes M$ is again $\mathfrak{Z}$-finite [BG, Corollary 2.6]. If we now tensor a module $M \in \mathcal{M}^\infty\big(\chi(\tau)\big)$ with E and then project onto the central character $\chi(\tau + \nu)$, we obtain the *translation functor*

$$\begin{array}{rccc} T_\tau^{\tau+\nu}: & \mathcal{M}^\infty\big(\chi(\tau)\big) & \longrightarrow & \mathcal{M}^\infty\big(\chi(\tau+\nu)\big) \\ & M & \mapsto & \mathrm{pr}_{\chi(\tau+\nu)}(E \otimes M) \end{array}$$

In the present paper we investigate the fine structure of translation functors. Namely, take for M the Verma module $M(\tau) := \mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{b})} k_\tau$ with its highest weight vector $v_\tau := 1 \otimes 1 \in M(\tau)_\tau$ and then identify for all $\tau \in \mathfrak{h}^*$:

$$\begin{array}{ccc} M(\tau) & \xrightarrow{\sim} & \mathfrak{U}(\mathfrak{n}^-) \\ uv_\tau & \mapsto & u \end{array}$$

We set $V(\nu) = V := E \otimes \mathfrak{U}(\mathfrak{n}^-)$, choose a fixed extremal weight vector $e_\nu \in E_\nu$ and define the map

$$\begin{array}{rccc} f_\nu: & \mathfrak{h}^* & \longrightarrow & V \\ & \tau & \mapsto & \mathrm{pr}_{\chi(\tau+\nu)}(e_\nu \otimes v_\tau) \end{array}$$

Here we identify $\mathrm{pr}_{\chi(\tau+\nu)}(e_\nu \otimes v_\tau) \in E \otimes M(\tau)$ with its image in V. The image of the map f_ν is then contained in the finite dimensional ν-weight space $(E \otimes \mathfrak{U}(\mathfrak{n}^-))_\nu$ of V, and we may thus regard f_ν as a map between varieties. In general however, f_ν is not a morphism.
Let $\rho := 1/2 \sum_{\alpha \in R^+} \alpha$ denote the half sum of positive roots, $\alpha^\vee$ the co-root of α and set

$$\mathcal{N}_\nu := \{(\alpha, m_\alpha) \in R^+ \times \mathbb{Z} \mid -\langle \rho, \alpha^\vee \rangle \leq m_\alpha < -\langle \nu + \rho, \alpha^\vee \rangle\}.$$

Then the set

$$\mathcal{H}_\nu := \bigcup_{(\alpha,m) \in \mathcal{N}_\nu} \{\tau \in \mathfrak{h}^* \mid \langle \tau, \alpha^\vee \rangle = m\}$$

is a finite family of hyperplanes and therefore Zariski closed. We will show at first that f_ν is a morphism of varieties on the complement of $\mathcal{H}_\nu \cup \mathcal{S}$, where $\mathcal{S} \subset \mathfrak{h}^*$ is a suitable Zariski closed subset of codimension ≥ 2. More precisely, $\mathcal{S}$ consists of intersections of finitely many hyperplanes.

Define for all roots $\alpha \in R$ and for all $m \in \mathbb{Z}$ the polynomial function $H_{\alpha,m}$ on $\mathfrak{h}^*$ by

$$H_{\alpha,m}(\tau) := \langle \tau, \alpha^\vee \rangle - m \text{ for all } \tau \in \mathfrak{h}^*.$$

If we then set $\delta_\nu := \prod_{(\alpha,m)\in\mathcal{N}_\nu} H_{\alpha,m}$, we obtain $\mathcal{H}_\nu = \{\tau \in \mathfrak{h}^* | \ \delta_\nu(\tau) = 0\}$. A central result of this paper will be the following

Theorem. *There exists a morphism of varieties $G : \mathfrak{h}^* \to V_\nu$, such that the set of zeros of G has codimension ≥ 2 and such that G equals $\delta_\nu f_\nu$ on $\mathfrak{h}^* - (\mathcal{H}_\nu \cup \mathcal{S})$.*

This means that the map f_ν has a pole of order 1 along each of the hyperplanes $\langle \tau, \alpha^\vee \rangle = m$ with $(\alpha, m) \in \mathcal{N}_\nu$ and outside of these hyperplanes it is a non-vanishing morphism of varieties except on a set of codimension ≥ 2.

In chapter 4 we introduce the so-called triangle functions $\Delta(\mu, \nu; x)(\tau)$ for integral weights μ and ν and $x \in \mathcal{W}$. These are rational functions on $\mathfrak{h}^*$, which measure in a subtle way the relation between the two translation functors $T_{\tau+\nu}^{\tau+\nu+\mu} \circ T_\tau^{\tau+\nu}$ and $T_\tau^{\tau+\nu+\mu}$ by first applying them to the Verma modules $M(\tau)$ and $M(x \cdot \tau)$ and then identifying the results with $M(\tau + \nu + \mu)$ and $M(x \cdot (\tau + \nu + \mu))$ respectively.
Bernstein defined in [Be] the so-called *relative trace* tr_E for a finite dimensional vector space E and this function is related to the special case $\Delta(-\nu, \nu; w_0)$, where ν is dominant and w_0 is the longest element of $\mathcal{W}$. This is explained in Chapter 5. By Bernstein's explicit formula for tr_E we obtain

Corollary. *Let $\nu \in P^+$ be dominant and $\tau \in \mathfrak{h}^*$ generic, i.e. $\langle \tau, \alpha^\vee \rangle \notin \mathbb{Z}$ for all $\alpha \in R$. Then*

$$\Delta(-\nu, \nu; w_0)(\tau) = \prod_{\alpha \in R^+} \frac{\langle \tau + \nu + \rho, \alpha^\vee \rangle}{\langle \tau + \rho, \alpha^\vee \rangle} \ .$$

We remark that Kashiwara has also calculated this case [Ka, Thm. 1.9].
In Chapter 6 we calculate the triangle functions in general. Since they are rational functions on $\mathfrak{h}^*$, it suffices to determine their zeros and poles. In order to do this we make use of the maps f_ν for suitable integral weights ν. Set now $\bar{\alpha}(\lambda) := 1$ for $\langle \lambda, \alpha^\vee \rangle < 0$ and $\bar{\alpha}(\lambda) := 0$ for $\langle \lambda, \alpha^\vee \rangle \geq 0$. Then we obtain

Theorem. *Let $\nu, \mu \in P$ be integral weights and $x \in \mathcal{W}$. Then*

$$\Delta(\mu, \nu; x)(\tau - \rho) = c \prod_{\substack{\alpha \in R^+ \\ \text{with } x\alpha \in R^-}} \frac{\langle \tau, \alpha^\vee \rangle^{\bar{\alpha}(\nu)} \langle \tau + \nu, \alpha^\vee \rangle^{\bar{\alpha}(\mu)} \langle \tau + \nu + \mu, \alpha^\vee \rangle^{\bar{\alpha}(\nu+\mu)}}{\langle \tau, \alpha^\vee \rangle^{\bar{\alpha}(\nu+\mu)} \langle \tau + \nu, \alpha^\vee \rangle^{\bar{\alpha}(\nu)} \langle \tau + \nu + \mu, \alpha^\vee \rangle^{\bar{\alpha}(\mu)}}$$

for a constant $c \in k^\times$ independent of τ, ν, μ and x.

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2. Filtration of $E \otimes M(\tau)$

In the following let $\nu \in P$ be a fixed integral weight. Then let $\nu_1, \dots, \nu_n$ be the multiset of weights of $E = E(\nu)$, i.e. with multiplicities, and let $(e_i)_{1 \le i \le n}$ be a basis of weight vectors of E, such that $e_i \in E_{\nu_i}$ and $\nu_i < \nu_j \Rightarrow i < j$. Here $\lambda \le \mu$ for weights $\lambda, \mu \in \mathfrak{h}^*$, if $\mu - \lambda = \sum_{\alpha \in R^+} n_\alpha \alpha$ with $n_\alpha \in \mathbb{N}$.
In [BGG1] it was shown that the tensor product $E \otimes M(\tau)$ admits a chain of submodules $N_i := \sum_{j=i}^n \mathfrak{U}(\mathfrak{n}^-)(e_j \otimes v_\tau)$, such that $N_i/N_{i+1} \cong M(\tau + \nu_i)$. Using this, one can easily construct a slightly coarser filtration, namely a filtration such that the subquotients are each isomorphic to a direct sum of Verma modules with the same highest weight. In order to do this, let $\mu_1, \mu_2, \dots, \mu_m$ denote the set of weights of E *without* multiplicities and set $d_j := \dim E_{\mu_j}$. We may choose the numbering of the ν_i in such a way that $\nu_k = \mu_j$ for all k with $d_1 + \cdots + d_{j-1} < k \le d_1 + \cdots + d_j$. The weight space E_{μ_j} is then generated by precisely these e_k. Set $d_0 := 0$ and define for $1 \le j \le m$:

$$M_j := \sum_{k=j}^{m} \mathfrak{U}(\mathfrak{n}^-)(E_{\mu_k} \otimes v_\tau) = \sum_{k > d_1 + d_2 + \cdots d_{j-1}}^{n} \mathfrak{U}(\mathfrak{n}^-)(e_k \otimes v_\tau).$$

We obtain immediately: The chain $E \otimes M(\tau) = M_1 \supset \cdots \supset M_m \supset 0$ is a filtration of submodules such that $M_j/M_{j+1} \cong \bigoplus_{d_j} M(\tau + \mu_j)$ and the summands $M(\tau + \mu_j)$ are generated by the vectors $(e_i \otimes v_\tau)/M_{j+1} = \mathrm{pr}(e_i \otimes v_\tau)/M_{j+1}$, where e_i is a vector of weight μ_j.

Lemma 1. *For $1 \le j \le m$ let $s_j \in \mathfrak{Z}$ with $\chi_{\tau+\mu_j}(s_j) = 0$ and set $z_i := \prod_{j \ge i} s_j$. Then for $1 \le i \le m$ we get: $z_i M_i = 0$.*
In particular, for a weight vector $e_{\mu_i} \in E_{\mu_i}$ we obtain

$$z_{i+1}(e_{\mu_i} \otimes v_\tau) = z_{i+1}\, \mathrm{pr}_{\chi(\tau+\mu_i)}(e_{\mu_i} \otimes v_\tau).$$

Proof. The first statement follows by induction from above and since a central element $z \in \mathfrak{Z}$ operates on the Verma module $M(\lambda)$ by multiplication with the scalar $\chi_\lambda(z)$.
By construction of the M_i it is clear, that for $e_{\mu_i} \otimes v_\tau \in M_i$ its projections $\mathrm{pr}_{\chi(\tau+\mu)}(e_{\mu_i} \otimes v_\tau)$ lie already in M_{i+1} if $\chi(\tau+\mu) \ne \chi(\tau+\mu_i)$. But $e_{\mu_i} \otimes v_\tau$ equals the sum of its projections and thus the second statement follows from the first. □

Note that for the extremal weight $\nu = \nu_{i_0} = \mu_{j_0}$ its weight space is of dimension 1. Therefore, we can always choose the numbering of the ν_i such that $i > i_0 \iff \nu_i > \nu_{i_0} = \nu$ or equivalently $j > j_0 \iff \mu_j > \mu_{j_0} = \nu$. It then follows immediately that an element of $M_{i_0} = M_{j_0}$ (of $M_{i_0+1} = M_{j_0+1}$ resp.) consists only of parts with central character $\chi_{\tau+\mu}$ for $\mu \in P(E)$ with $\mu \ge \nu$ ($\mu > \nu$ resp.). For $e_\nu \otimes v_\tau \in M_{i_0}$ we thus obtain the following special case of Lemma 1

Lemma 2. *For $\mu \in P(E)$ with $\mu > \nu$ let $s_\mu \in \mathfrak{Z}$ such that $\chi_{\tau+\mu}(s_\mu) = 0$ and set $z := \prod_{\mu > \nu} s_\mu$. Then $z(e_\nu \otimes v_\tau) = z\, \mathrm{pr}_{\chi(\tau+\nu)}(e_\nu \otimes v_\tau)$.*

3. The fine structure of $E \otimes M(\tau)$

3.1. **Algebraicity of f_ν.** For $\nu \in P$ fixed, set $E := E(\nu)$ and let f_ν be the map

$$f_\nu : \begin{array}{ccc} \mathfrak{h}^* & \longrightarrow & V \\ \tau & \mapsto & \mathrm{pr}_{\chi(\tau+\nu)}(e_\nu \otimes v_\tau) \end{array}$$

where $V = V(\nu) := E \otimes \mathfrak{U}(\mathfrak{n}^-) \cong E \otimes M(\tau)$ for all $\tau \in \mathfrak{h}^*$. We will first construct a Zariski open set $\mathcal{U} \subset \mathfrak{h}^*$ such that f_ν restricted to $\mathcal{U}$ is a morphism of varieties. In this case we will also say that f_ν is *algebraic* on $\mathcal{U}$.

Before we go on, we need some more notation : The dot-operation of the Weyl group on $\mathfrak{h}^*$ has fixed point $-\rho = -1/2 \sum_{\alpha \in R^+} \alpha$ and is defined by $w \cdot \lambda := w(\lambda + \rho) - \rho$. Let $\mathcal{P}(\mathfrak{h}^*)$ denote the set of polynomial functions on $\mathfrak{h}^*$ and $\mathcal{P}(\mathfrak{h}^*)^{\mathcal{W}\cdot}$ the $(\mathcal{W}\cdot)$-invariants. We define the operator

$$\mathrm{sym} : \mathcal{P}(\mathfrak{h}^*) \longrightarrow \mathcal{P}(\mathfrak{h}^*)^{\mathcal{W}\cdot}$$

by $(\mathrm{sym}\, s)(\lambda) := \prod_{x \in \mathcal{W}} s(x \cdot \lambda) \quad$ for all $\lambda \in \mathfrak{h}^*, s \in \mathcal{P}(\mathfrak{h}^*)$.

For any $\mu \in P(E)$ with $\mu > \nu$ we now choose an element $h_\mu \in \mathfrak{h}$, $h_\mu \neq 0$ and define maps $H_\mu : \mathfrak{h}^* \times \mathfrak{h}^* \longrightarrow k$ by

$$H_\mu(\lambda, \tau) := \langle \lambda - \tau - \mu, h_\mu \rangle \quad \text{for all } \lambda, \tau \in \mathfrak{h}^*.$$

If we fix $\tau \in \mathfrak{h}^*$, the map $H_\mu(-, \tau) : \mathfrak{h}^* \longrightarrow k$ is then a polynomial function on $\mathfrak{h}^*$ and its kernel is obviously a hyperplane in $\mathfrak{h}^*$. We then define a $(\mathcal{W}\cdot)$-invariant polynomial function p_τ, depending also on the choice of the elements h_μ, by

$$p_\tau := \prod_{\mu \in P(E), \mu > \nu} \mathrm{sym}\, H_\mu(-, \tau).$$

By means of the Harish-Chandra isomorphism [Di, 7.4] $\xi : \mathfrak{Z} \xrightarrow{\sim} \mathcal{P}(\mathfrak{h}^*)^{\mathcal{W}\cdot}$, there exist central elements $s_\mu \in \mathfrak{Z}$, such that $\xi(s_\mu) = \mathrm{sym}\, H_\mu(-, \tau)$. For $z_\tau := \prod_{\mu > \nu} s_\mu$ we have $\xi(z_\tau) = p_\tau$. Define now the map

$$u = u_{\{h_\mu\}} : \begin{array}{ccc} \mathfrak{h}^* & \longrightarrow & k \\ \tau & \mapsto & p_\tau(\tau + \nu) \end{array}$$

and set $\mathcal{U}_{\{h_\mu\}} := \{\tau \in \mathfrak{h}^* \mid u(\tau) \neq 0\} \subset \mathfrak{h}^*$.

Lemma 3. *The map f_ν restricted to $\mathcal{U}_{\{h_\mu\}}$ is a morphism of varieties and for all $\tau \in \mathcal{U}_{\{h_\mu\}}$ we have $f_\nu(\tau) = (u(\tau))^{-1} z_\tau(e_\nu \otimes v_\tau)$.*

Proof. We know $\chi_{\tau+\mu}(s_\mu) = (\xi(s_\mu))(\tau + \mu) = (\mathrm{sym}\, H_\mu(-, \tau))(\tau + \mu) = 0$ and by Lemma 2 we conclude $z_\tau(e_\nu \otimes v_\tau) = z_\tau\, \mathrm{pr}_{\chi(\tau+\nu)}(e_\nu \otimes v_\tau)$.

Let now $\tau \in \mathcal{U}_{\{h_\mu\}}$, i. e. $u(\tau) \neq 0$. This means, that for all $w \in \mathcal{W}$ and for all $\mu \in P(E)$ with $\mu > \nu$ we have $\langle w \cdot (\tau + \nu) - \tau - \mu, h_\mu \rangle \neq 0$ and in particular $w \cdot (\tau + \nu) \neq \tau + \mu$ or equivalently $\chi(\tau + \nu) \neq \chi(\tau + \mu) \forall \mu \in P(E), \mu > \nu$. This implies, that already the projection $\mathrm{pr}_{\chi(\tau+\nu)}(e_\nu \otimes v_\tau) \in E \otimes M(\tau)$ generates a Verma module $M(\tau + \nu)$. A central element z operates on this by multiplication with the scalar $(\xi(z))(\tau + \nu) = u(\tau)$. We therefore get $z_\tau(e_\nu \otimes v_\tau) = z_\tau\, \mathrm{pr}_{\chi(\tau+\nu)}(e_\nu \otimes v_\tau) = u(\tau) f_\nu(\tau)$ and the above equation follows.

We have yet to show that f_ν is algebraic on $\mathcal{U}_{\{h_\mu\}}$. By construction of z_τ it is clear that z_τ depends algebraically on τ for all $\tau \in \mathfrak{h}^*$. Also the map $\tau \mapsto z_\tau(e_\nu \otimes v_\tau) \in V$ is a morphism on $\mathfrak{h}^*$ and $1/u$ is per definition algebraic on $\mathcal{U}_{\{h_\mu\}}$. Thus f_ν restricted to $\mathcal{U}_{\{h_\mu\}}$ is a morphism. □

Example. *Let $\nu \in P(E)$ be an extremal and dominant weight. For all $\mu \in P(E)$ with $\mu \neq \nu$ we then have $\mu < \nu$ and we may choose $z_\tau := 1 \in \mathfrak{Z}$ for all $\tau \in \mathfrak{h}^*$. The above lemma then implies $\mathcal{U}_{\{h_\mu\}} = \mathfrak{h}^*$, the map f_ν is a morphism on $\mathfrak{h}^*$ and $f_\nu(\tau) = e_\nu \otimes v_\tau$ for all τ.*

Lemma 3 implies that f_ν is algebraic on $\mathcal{U} := \bigcup \mathcal{U}_{\{h_\mu\}}$, where the union is taken over all possible choices for $\{h_\mu \mid \mu \in P(E), \mu > \nu\}$. If we set

$$\mathcal{A} := \{\tau \in \mathfrak{h}^* \mid u(\tau) = p_\tau(\tau+\nu) = 0 \text{ for all choices of } \{h_\mu\}\},$$

we can write $\mathcal{U}$ as $\mathcal{U} = \mathfrak{h}^* - \mathcal{A}$. Since $p_\tau(\tau+\nu) = \prod_{\mu \in P(E), \mu > \nu} \prod_{w \in \mathcal{W}} \langle w \cdot (\tau+\nu) - \tau - \mu, h_\mu \rangle$ we know that τ is in $\mathcal{A}$ if and only if there exists a $w \in \mathcal{W}$ and a $\mu \in P(E)$ with $\mu > \nu$, such that $w \cdot (\tau+\nu) = \tau + \mu$. Thus we obtain

$$\mathcal{A} = \bigcup_{\substack{\mu \in P(E), \mu > \nu \\ w \in \mathcal{W}}} \{\tau \in \mathfrak{h}^* \mid w \cdot (\tau+\nu) = \tau + \mu\}.$$

Let us examing more closely the sets on the right hand side: For $w = s_\alpha, \alpha \in R$, we get $\{\tau \in \mathfrak{h}^* \mid \langle \tau, \alpha^\vee \rangle \alpha = \nu - \mu - \langle \nu + \rho, \alpha^\vee \rangle \alpha\}$, which implies that this set is nonempty if and only if there exists a weight $\mu = \nu - \langle \tau + \nu + \rho, \alpha^\vee \rangle \alpha$ in the α-string through ν, such that $\mu \in P(E)$ and $\mu > \nu$. Since ν was an extremal weight, it is either the greatest or the smallest weight in this α-string and hence such a μ exists only if $\langle \nu, \alpha^\vee \rangle < 0$ for an $\alpha \in R^+$. In this case all $\mu_{(n)} := \nu + n\alpha$ for $1 \leq n \leq -\langle \nu, \alpha^\vee \rangle$ are weights of E with $\mu_{(n)} \geq \nu$. Comparing this with $\mu = \nu - \langle \nu + \rho + \tau, \alpha^\vee \rangle \alpha$ we obtain as condition for τ precisely $-\langle \rho, \alpha^\vee \rangle \leq \langle \tau, \alpha^\vee \rangle \leq -\langle \nu + \rho, \alpha^\vee \rangle - 1$ for an $\alpha \in R^+$.
Set $\mathcal{W}^0 := \{w \in \mathcal{W} \mid w \neq s_\alpha \text{ for an } \alpha \in R\}$. For

$$\begin{aligned}
\mathcal{N}_\nu &:= \{(\alpha, m_\alpha) \in R^+ \times \mathbb{Z} \mid -\langle \rho, \alpha^\vee \rangle \leq m_\alpha < -\langle \nu + \rho, \alpha^\vee \rangle\} \\
\mathcal{H}_\nu &:= \bigcup_{(\alpha, m) \in \mathcal{N}_\nu} \{\tau \in \mathfrak{h}^* \mid \langle \tau, \alpha^\vee \rangle = m\} \\
\mathcal{S} &:= \bigcup_{\substack{\mu \in P(E), \mu > \nu \\ w \in \mathcal{W}^0}} \{\tau \in \mathfrak{h}^* \mid w \cdot (\tau+\nu) = \tau + \mu\},
\end{aligned}$$

we get $\mathcal{A} = \mathcal{H}_\nu \cup \mathcal{S}$ and $\mathcal{S}$ consists of finitely many intersections of at least two hyperplanes and is therefore a Zariski closed subset of codimension ≥ 2. We remark that in general $\mathcal{H}_\nu$ and $\mathcal{S}$ are not disjoint. Using Lemma 3 we obtain

Lemma 4. *The map f_ν is algebraic on the complement of $\mathcal{H}_\nu \cup \mathcal{S}$.*

3.2. **Poles of f_ν. Theorem 1 and proof.** Let now $\delta_\nu \in \mathcal{P}(\mathfrak{h}^*)$ be the product of the equations of the hyperplanes in $\mathcal{H}_\nu$, that is

$$\delta_\nu := \prod_{(\alpha, m) \in \mathcal{N}_\nu} H_{\alpha, m} = \prod_{\alpha \in R^+} \prod_{0 \leq n_\alpha < -\langle \nu, \alpha^\vee \rangle} \left(\langle \tau + \rho, \alpha^\vee \rangle - n_\alpha \right),$$

where $H_{\alpha,m}(\tau) := \langle \tau, \alpha^\vee \rangle - m$. We then have $\mathcal{H}_\nu = \{\tau \in \mathfrak{h}^* \mid \delta_\nu(\tau) = 0\}$.

Theorem 1. *There exists a morphism of varieties $G : \mathfrak{h}^* \longrightarrow V_\nu$, such that the set of zeros of G has codimension ≥ 2 and such that G equals $\delta_\nu f_\nu$ on $\mathfrak{h}^* - (\mathcal{H}_\nu \cup \mathcal{S})$.*

The proof comes in two parts. In the first part we demonstrate the existence of an algebraic extension G of $\delta_\nu f_\nu$ on the whole of $\mathfrak{h}^*$, in the second we will show that the set of zeros of G has codimension ≥ 2. It will be useful to introduce the set $\mathcal{S}_1 \subset \mathfrak{h}^*$ of all intersections of hyperplanes in $\mathcal{H}_\nu$. So if we set

$$\mathcal{S}_1 := \{\tau \in \mathfrak{h}^* \mid H_{\alpha,m}(\tau) = 0 = H_{\beta,n}(\tau) \text{ for } (\alpha, m) \neq (\beta, n) \in \mathcal{N}_\nu\}$$

then $\mathcal{S} \cup \mathcal{S}_1$ is a Zariski closed subset of codimension ≥ 2. Note that $(\mathcal{H}_\nu \cap \mathcal{S}) \subset \mathcal{S}_1$ and hence $\mathcal{H}_\nu - (\mathcal{S} \cup \mathcal{S}_1) = \mathcal{H}_\nu - \mathcal{S}_1$. By definition of $\mathcal{S}_1$ we obtain immediately

Lemma 5. *Let $\tau \in \mathcal{H}_\nu - \mathcal{S}_1$. Then there exists exactly one $\alpha \in R^+$, such that $s_\alpha \cdot (\tau + \nu) = \tau + \mu_0$ for a weight μ_0 of E with $\mu_0 > \nu$, namely $\mu_0 = \nu - \langle \tau + \nu + \rho, \alpha^\vee \rangle \alpha$. For all $\mu \in P(E)$ with $\mu > \nu$ and $\mu \neq \mu_0$ we have $w \cdot (\tau + \nu) \neq \tau + \mu$ for all $w \in \mathcal{W}$.*

Proof of Theorem 1. Again let $\mathcal{U} = \mathfrak{h}^* - (\mathcal{H}_\nu \cup \mathcal{S})$ be the set on which f_ν is algebraic. First we claim

(∗) For all $\tau_0 \in \mathfrak{h}^* - (\mathcal{S} \cup \mathcal{S}_1)$ there exists a Zariski open neighborhood $\mathcal{U}_0$ of τ_0 and an algebraic map $G_0 : \mathcal{U}_0 \to V_\nu$, such that $G_0 = \delta_\nu f_\nu$ on $\mathcal{U} \cap \mathcal{U}_0$.

This statement then implies the existence of an algebraic extension of $\delta_\nu f_\nu$ *locally* around every point of $\mathfrak{h}^* - (\mathcal{S} \cup \mathcal{S}_1)$. This extension is unique and thus we obtain a *global* algebraic extension on $\mathfrak{h}^* - (\mathcal{S} \cup \mathcal{S}_1)$. Since $\operatorname{codim}(\mathcal{S} \cup \mathcal{S}_1) \geq 2$ we can extend this to a morphism G on the whole of $\mathfrak{h}^*$.

Let now $\tau_0 \in \mathfrak{h}^* - (\mathcal{S} \cup \mathcal{S}_1)$.

If $\tau_0 \notin \mathcal{H}_\nu$, then $\tau_0 \in \mathcal{U} = \mathfrak{h}^* - (\mathcal{H}_\nu \cup \mathcal{S})$ and since f_ν was algebraic on $\mathcal{U}$ (Lemma 4), claim (∗) follows with $\mathcal{U}_0 := \mathcal{U}$ and $G_0 := \delta_\nu f_\nu$.

Let now $\tau_0 \in \mathcal{H}_\nu$. By assumption we then have $\tau_0 \in \mathcal{H}_\nu - (\mathcal{S} \cup \mathcal{S}_1) = \mathcal{H}_\nu - \mathcal{S}_1$, and Lemma 5 implies that $\langle \tau_0, \alpha^\vee \rangle = t_0 \in \mathbb{Z}$ for exactly one $\alpha \in R^+$ and $\mu_0 = \nu - \langle \nu + \rho + \tau_0, \alpha^\vee \rangle \alpha$ is a weight of E with $\mu_0 > \nu$.

Again we construct a set $\mathcal{U}_{\{h_\mu\}}$ with a particular choice of $\{h_\mu\}$. Namely, choose for all weights $\mu \in P(E)$ with $\mu > \nu$ and $\mu \neq \mu_0$ elements $h_\mu \in \mathfrak{h}$, such that $H_\mu(w \cdot (\tau_0 + \nu), \tau_0) = \langle w \cdot (\tau_0 + \nu) - \tau_0 - \mu, h_\mu \rangle \neq 0$ for all $w \in \mathcal{W}$. For μ_0 however, choose h_{μ_0} such that

a) $\langle \alpha, h_{\mu_0} \rangle \neq 0$ and
b) $H_{\mu_0}(w \cdot (\tau_0 + \nu), \tau_0) = \langle w \cdot (\tau_0 + \nu) - \tau_0 - \mu_0, h_{\mu_0} \rangle \neq 0$ for all $s_\alpha \neq w \in \mathcal{W}$.

Lemma 5 ensures that such a choice of $\{h_\mu\}$ is always possible. For $\tau \in \mathfrak{h}^*$ we set again $p_\tau := \prod_{\mu \in P(E), \mu > \nu} \operatorname{sym} H_\mu(-, \tau)$, and $\xi^{-1}(p_\tau) =: z_\tau \in \mathfrak{Z}$. For u defined by $u(\tau) = p_\tau(\tau + \nu)$ we then obtain $\mathcal{U}_{\{h_\mu\}} = \{\tau \in \mathfrak{h}^* \mid u(\tau) \neq 0\}$. According to Lemma 3 the map f_ν restricted to $\mathcal{U}_{\{h_\mu\}}$ is algebraic. Let's have a closer look at the map u : It vanishes along the hyperplane $\ker H_{\alpha,t_0}$, because for all $\tau \in \mathfrak{h}^*$ we have $H_{\mu_0}\big(s_\alpha \cdot (\tau + \nu), \tau\big) = -\langle \alpha, h_{\mu_0} \rangle \cdot \big(\langle \tau, \alpha^\vee \rangle - t_0\big) = a \cdot H_{\alpha,t_0}(\tau)$, where $a := -\langle \alpha, h_{\mu_0} \rangle \neq 0$ according to assumption a).

All the other hyperplanes along which u vanishes are not equal to $\ker H_{\alpha,t_0}$ but do at most intersect it. Otherwise (since $\tau_0 \in \ker H_{\alpha,t_0}$) we would have

$H_\mu\big(w \cdot (\tau_0 + \nu), \tau_0\big) = 0$ for a pair $(\mu, w) \neq (\mu_0, s_\alpha)$ which is impossible by our choice of the h_μ.
For $\bar{u} : \mathfrak{h}^* \to k$ defined by $\bar{u} H_{\alpha,t_0} = u$, it follows by what we just said that $\bar{u}(\tau_0) \neq 0$. With $\mathcal{U}_0 := \{\tau \in \mathfrak{h}^* \mid \bar{u}(\tau) \neq 0\}$ we then get

- $\tau_0 \in \mathcal{U}_0$
- $1/\bar{u}$ is algebraic on $\mathcal{U}_0$
- $\mathcal{U}_{\{h_\mu\}} = \mathcal{U}_0 - \ker H_{\alpha,t_0}$, thus $\mathcal{U}_0 = \mathcal{U}_{\{h_\mu\}} \cup \ker H_{\alpha,t_0}$
- $(\mathcal{U} \cap \mathcal{U}_0) \subset \mathcal{U}_{\{h_\mu\}}$, since $\mathcal{U} \cap \ker H_{\alpha,t_0} = \emptyset$.

If we define $\bar{\delta} = \bar{\delta}_\nu : \mathfrak{h}^* \to k$ by $\bar{\delta} H_{\alpha,t_0} = \delta_\nu$ and set

$$\begin{array}{rccc} G_0 : & \mathcal{U}_0 & \longrightarrow & V \\ & \tau & \mapsto & \bar{\delta}(\tau)\big(\bar{u}(\tau)\big)^{-1} z_\tau(e_\nu \otimes v_\tau) \end{array}$$

we know that G_0 is algebraic on $\mathcal{U}_0$. In particular, G_0 is algebraic on an open neighborhood of τ_0. This is now the local algebraic extension of $\delta_\nu f_\nu$ around τ_0 which we were looking for, because according to Lemma 3 we have for all $\tau \in \mathcal{U}_{\{h_\mu\}}$:

$$\begin{aligned} \delta_\nu(\tau) f_\nu(\tau) &= \delta_\nu(\tau)\,(u(\tau))^{-1}\, z_\tau(e_\nu \otimes v_\tau) \\ &= \bar{\delta}_\nu(\tau) H_{\alpha,t_0}(\tau)\,(H_{\alpha,t_0}(\tau))^{-1}\,(\bar{u}(\tau))^{-1}\, z_\tau(e_\nu \otimes v_\tau) \\ &= G_0(\tau). \end{aligned}$$

In particular, this equation holds for all $\tau \in (\mathcal{U} \cap \mathcal{U}_0) \subset \mathcal{U}_{\{h_\mu\}}$, i.e. $(*)$.

It remains to show that the set of zeros of G has codimension ≥ 2.
In order to do this, let $\mathcal{S}_2$ be the union of all intersections of hyperplanes in $\mathcal{H}_\nu$ with any other integral hyperplanes, that is

$$\mathcal{S}_2 := \mathcal{S}_1 \cup \{\tau \in \mathcal{H}_\nu \mid \langle \tau, \alpha^\vee \rangle = m \in \mathbb{Z} \text{ for a pair } (\alpha, m) \notin \mathcal{N}_\nu\}.$$

We claim

$(**)$ $\mathcal{K} := \{\tau \in \mathfrak{h}^* \mid G(\tau) = 0\} \subset (\mathcal{S} \cup \mathcal{S}_2)$

and conclude then that $\operatorname{codim} \mathcal{K} \geq 2$. Indeed, by definition the set $\mathcal{S} \cup \mathcal{S}_2$ consists of a countable family of intersections of integral hyperplanes. On the other hand, G is algebraic on $\mathfrak{h}^*$ and therefore its set of zeros must be Zariski closed. Since a countable family of intersections of integral hyperplanes is Zariski closed only if it is a *finite* family of such intersections, the set $\mathcal{K}$ must have codimension ≥ 2.
Let now $\tau_0 \in \mathfrak{h}^* - (\mathcal{S} \cup \mathcal{S}_2)$. We have to show $G(\tau_0) \neq 0$.
If $\tau_0 \notin \mathcal{H}_\nu$, we have $\delta_\nu(\tau_0) \neq 0$ and $\tau_0 \in \mathcal{U} = \mathfrak{h}^* - (\mathcal{H}_\nu \cup \mathcal{S})$. Now G equals $\delta_\nu f_\nu$ on $\mathcal{U}$ and hence $G(\tau_0) = \delta_\nu(\tau_0) f_\nu(\tau_0) \neq 0$.
Let now $\tau_0 \in \mathcal{H}_\nu$. Again by the first part of the proof we know that there is an open neighborhood $\mathcal{U}_0$ of τ_0, such that $G(\tau) = \bar{\delta}_\nu(\tau)\big(\bar{u}(\tau)\big)^{-1} z_\tau(e_\nu \otimes v_\tau)$ for all $\tau \in \mathcal{U}_0$. Since $\bar{\delta}_\nu(\tau_0) \neq 0$ it thus suffices to show

$(**)$' $z_{\tau_0}(e_\nu \otimes v_{\tau_0}) \neq 0$.

In order to do this, we will use Lemma 6 to 9 of the following section. Set $e := e_\nu$ and denote the projection onto the central character $\chi(\tau_0 + \nu)$ by $\mathrm{pr} = \mathrm{pr}_{\chi(\tau_0+\nu)} : V \twoheadrightarrow V$.
Since $\tau_0 \in \mathcal{H}_\nu - \mathcal{S}_2$, there exists exactly one $\alpha \in R^+$ such that $\langle \tau_0, \alpha^\vee \rangle \in \mathbb{Z}$. Therefore, $R_{\tau_0} := \{\beta \in R \mid \langle \tau_0, \beta^\vee \rangle \in \mathbb{Z}\} = \{\alpha, -\alpha\}$ and hence the Weyl

group $\mathcal{W}_{\tau_0}$ of this root system R_{τ_0} equals $\mathcal{W}_{\tau_0} = \langle s_\beta \mid \beta \in R_{\tau_0}\rangle = \langle s_\alpha\rangle$. For τ_0 and $\mu_0 = \nu - \langle \tau_0 + \nu + \rho, \alpha^\vee\rangle\alpha$ we may then apply Lemma 8 and obtain a short exact sequence

$$\bigoplus_{\dim E_{\mu_0}} M(s_\alpha \cdot (\tau_0+\nu)) \hookrightarrow \mathrm{pr}\big(E \otimes M(\tau_0)\big) \twoheadrightarrow L(\tau_0+\nu)$$

such that $\mathrm{pr}(e \otimes v_{\tau_0})/\,\mathrm{pr}\big(E \otimes M(\tau_0)\big)$ generates the simple module $L(\tau_0+\nu)$. The module $\mathrm{pr}\big(E \otimes M(\tau_0)\big)$ is projective (Lemma 7) and this forces the above short exact sequence to be a nontrivial extension because otherwise, being a direct summand, $L(\tau_0+\nu)$ would be projective too. This contradicts Lemma 7.

Therefore (see for example [HS, Ch.3, Lemma 4.1]) we have at least one direct summand $M(s_\alpha \cdot (\tau_0+\nu))$ of $R_2 := \bigoplus_{\dim E_{\mu_0}} M(s_\alpha \cdot (\tau_0+\nu))$, such that the projection $\mathrm{pro} : R_2 \twoheadrightarrow M(s_\alpha \cdot (\tau_0+\nu))$ induces a nontrivial extension

$$M(s_\alpha \cdot (\tau_0+\nu)) \hookrightarrow \mathrm{PO} \twoheadrightarrow L(\tau_0+\nu)$$

together with a homomorphism $\tilde{\mathrm{pro}} : R_1 := \mathrm{pr}\big(E \otimes M(\tau_0)\big) \to \mathrm{PO}$, such that the following diagram commutes (the rows are exact, PO is a push-out of $\mathrm{pro} : R_2 \twoheadrightarrow M(s_\alpha \cdot (\tau_0+\nu))$ and $R_2 \hookrightarrow R_1$, see for example the dual statement to [HS, Ch.3, Lemma 1.3]):

$$\begin{tikzcd}
R_2 \arrow[r] \arrow[d, "\mathrm{pro}"] & R_1 \arrow[r] \arrow[d, "\tilde{\mathrm{pro}}"] & R_1/R_2 \cong L(\tau_0+\nu) \arrow[d, "\mathrm{id}"] \\
M(s_\alpha \cdot (\tau_0+\nu)) \arrow[r] & \mathrm{PO} \arrow[r] & \mathrm{PO}/M(s_\alpha \cdot (\tau_0+\nu)) \cong L(\tau_0+\nu)
\end{tikzcd}$$

Since the bottom row is a nontrivial extension and since $\mathcal{W}_{\tau_0+\nu} = \mathcal{W}_{\tau_0} = \langle s_\alpha\rangle$, the push-out PO is isomorphic to $P(\tau_0+\nu)$, the projective cover of $L(\tau_0+\nu)$ in $\mathcal{O}$ (Lemma 6). In the right loop of the above diagram lies the following element diagram

$$\begin{tikzcd}
\mathrm{pr}(e \otimes v_{\tau_0}) \arrow[r, mapsto] \arrow[d, mapsto] & \mathrm{pr}(e \otimes v_{\tau_0})/R_2 \arrow[d, mapsto] \\
\tilde{\mathrm{pro}}\big(\mathrm{pr}(e \otimes v_{\tau_0})\big) \arrow[r, mapsto] & \tilde{\mathrm{pro}}\big(\mathrm{pr}(e \otimes v_{\tau_0})\big)/M(s_\alpha \cdot (\tau_0+\nu))
\end{tikzcd}$$

Since $\mathrm{pr}(e \otimes v_{\tau_0})/R_2$ is a generator of $L(\tau_0+\nu)$ we conclude that $\tilde{\mathrm{pro}}\big(\mathrm{pr}(e \otimes v_{\tau_0})\big)/M(s_\alpha \cdot (\tau_0+\nu)) \in \mathrm{PO}/M(s_\alpha \cdot (\tau_0+\nu))$ is also a generator of $L(\tau_0+\nu)$ and thus its preimage $\tilde{\mathrm{pro}}\big(\mathrm{pr}(e \otimes v_{\tau_0})\big)$ generates the indecomposable module $\mathrm{PO} \cong P(\tau_0+\nu)$. By Lemma 9 the element z_{τ_0} is not contained in the annihilator of $P(\tau_0+\nu)$, this means in particular $0 \neq z_{\tau_0}\tilde{\mathrm{pro}}\big(\mathrm{pr}(e \otimes v_{\tau_0})\big) = \tilde{\mathrm{pro}}\big(z_{\tau_0}\,\mathrm{pr}(e \otimes v_{\tau_0})\big)$ and we obtain $z_{\tau_0}\,\mathrm{pr}(e \otimes v_{\tau_0}) \neq 0$. Therefore, claim $(**)$' is proven and the proof of Theorem 1 is finished provided we know that Lemma 6, 7, 8 and 9 hold. $\square$

3.3. **Proof of Lemma 6, 7, 8 and 9.** In the following let e_ν denote again the fixed extremal weight vector of the finite dimensional irreducible $\mathfrak{g}$-module $E = E(\nu)$. Set $v_\tau \in M(\tau)$ the canonical generator of the Verma module and let $V := E \otimes \mathfrak{U}(\mathfrak{n}^-) \cong E \otimes M(\tau)$ for all $\tau \in \mathfrak{h}^*$. The category $\mathcal{O}$ is the category of all finitely generated $\mathfrak{g}$-modules, which are $\mathfrak{b}$-finite and

semisimple over $\mathfrak{h}$ [BGG2]. For $\lambda \in \mathfrak{h}^*$ set $\mathcal{W}_\lambda := \{w \in \mathcal{W} \mid \lambda - w\lambda \in P\}$ and $R_\lambda := \{\beta \in R \mid \langle \lambda, \beta^\vee \rangle \in \mathbb{Z}\}$. The group $\mathcal{W}_\lambda$ is then the Weyl group to the root system R_λ [Ja1, 1.3]. Denote by $P(\lambda)$ the projective cover in $\mathcal{O}$ of the simple module $L(\lambda)$.

Lemma 6. *Let $\lambda \in \mathfrak{h}^*$ with $\mathcal{W}_\lambda = \langle s_\alpha \rangle$ and $M(\lambda) = L(\lambda)$ simple. If the short exact sequence $M(s_\alpha \cdot \lambda) \hookrightarrow N \twoheadrightarrow L(\lambda)$ is a nontrivial extension, then N is isomorphic to $P(\lambda)$.*

Proof. For λ with $\mathcal{W}_\lambda = \langle s_\alpha \rangle$ there exists in $\mathcal{O}$ (up to isomorphism) a unique indecomposable projective module $P(\lambda)$, such that the short exact sequence

$$M(s_\alpha \cdot \lambda) \hookrightarrow P(\lambda) \twoheadrightarrow L(\lambda)$$

is a nontrivial extension [BGG2, Ch.4]. We thus have a nontrivial element of $\mathrm{Ext}^1\big(L(\lambda), M(s_\alpha \cdot \lambda)\big)$. The assertion follows immediately if we knew that $\dim \mathrm{Ext}^1\big(L(\lambda), M(s_\alpha \cdot \lambda)\big) \le 1$. Indeed, construct for $M(s_\alpha \cdot \lambda) \hookrightarrow P(\lambda) \twoheadrightarrow L(\lambda)$ the long exact homology sequence [HS, Ch.3, Thm.5.3] (set $M = M(s_\alpha \cdot \lambda), P = P(\lambda), L = L(\lambda), \mathrm{Ext} = \mathrm{Ext}_{\mathcal{O}}$ and let ω be the connecting homomorphism):

$$\cdots \to \mathrm{Hom}(M, M) \xrightarrow{\omega} \mathrm{Ext}^1(L, M) \to \mathrm{Ext}^1(P, M) \to \mathrm{Ext}^1(M, M) \to \cdots$$

and note that $\mathrm{Ext}^1(P, M) = 0$ since $P = P(\lambda)$ is projective [HS, Chap.3, Prop.2.6.]. By exactness we obtain ω surjective. Since now for any Verma module M $\dim \mathrm{Hom}(M, M) = 1$, we conclude that $\dim \mathrm{Ext}^1(L, M) \le 1$. $\square$

Lemma 7. *Let $\tau \in \mathfrak{h}^*$ with $\mathcal{W}_\tau = \langle s_\alpha \rangle$.*

(a) *If $\langle \tau + \rho, \alpha^\vee \rangle \ge 0$, then $\mathrm{pr}_{\chi(\tau+\nu)}\big(E \otimes M(\tau)\big)$ is projective, this means $\mathrm{pr}_{\chi(\tau+\nu)}\big(E \otimes M(\tau)\big)$ is a projective object of the category $\mathcal{O}$.*

(b) *If $\langle \tau + \nu + \rho, \alpha^\vee \rangle \le 0$, then $M(\tau + \nu) = L(\tau + \nu)$ is simple and not projective.*

Proof. a) A Verma module $M(\lambda)$ is projective if $\langle \lambda + \rho, \beta^\vee \rangle \ge 0$ for all $\beta \in R_\lambda \cap R^+$ [Ja2, 4.8]. Since $\mathcal{W}_\tau = \langle s_\alpha \rangle$, we have $R_\tau \cap R^+ = \{\alpha\}$ and hence $M(\tau)$ is projective. Then also $E \otimes M(\tau)$ is projective, because for E with $\dim E < \infty$ the functor

$$F_E : \begin{array}{ccc} \mathcal{O} & \longrightarrow & \mathcal{O} \\ M & \mapsto & E \otimes M \end{array}$$

maps projective objects of $\mathcal{O}$ to projective objects of $\mathcal{O}$ [BG]. Its direct summand $\mathrm{pr}_{\chi(\tau+\nu)}\big(E \otimes M(\tau)\big) \subset E \otimes M(\tau)$ is then projective too.

b) A Verma module $M(\lambda)$ is simple if and only if $\langle \lambda + \rho, \beta^\vee \rangle \le 0$ for all $\beta \in R_\lambda \cap R^+$ [Di, 7.6.24] or [Ja1, 1.8, 1.9]. Since $\mathcal{W}_{\tau+\nu} = \mathcal{W}_\tau = \langle s_\alpha \rangle$ we obtain $R_{\tau+\nu} \cap R^+ = \{\alpha\}$ and the simplicity of $M(\tau + \nu) = L(\tau + \nu)$ follows. Since the short exact sequence $M(s_\alpha \cdot (\tau + \nu)) \hookrightarrow P(\tau + \nu) \twoheadrightarrow L(\tau + \nu)$ is a nontrivial extension [BGG2, Ch.4], the module $L(\tau + \nu)$ cannot be projective. $\square$

Lemma 8. *Let $\tau \in \mathfrak{h}^*$ with $\mathcal{W}_\tau = \langle s_\alpha \rangle$ and $\mu_0 := \nu - \langle \tau+\nu+\rho, \alpha^\vee \rangle \alpha \in P(E)$ such that $\mu_0 > \nu$. Then the module $\operatorname{pr}_{\chi(\tau+\nu)} \big(E \otimes M(\tau)\big)$ admits a chain of submodules*

$$\operatorname{pr}_{\chi(\tau+\nu)} \big(E \otimes M(\tau)\big) = R_1 \supset R_2 \supset 0,$$

such that $R_2 \cong \bigoplus_{\dim E_{\mu_0}} M(s_\alpha \cdot (\tau+\nu))$ and $R_1/R_2 \cong M(\tau+\nu) = L(\tau+\nu)$ and such that $L(\tau+\nu)$ is generated by $\big(\operatorname{pr}_{\chi(\tau+\nu)}(e_\nu \otimes v_\tau)\big)/R_2$.

Proof. Let $\mu_1, \mu_2, \dots, \mu_m$ be the weights of E without multiplicities, numbered such that $\mu_i < \mu_j \Rightarrow i < j$. Set $\mu_{j_0} := \nu, \mu_{j_1} := \mu_0$, $d_j := \dim E_{\mu_j}$ and $\operatorname{pr} := \operatorname{pr}_{\chi(\tau+\nu)}$. From chapter 2 we already know that $E \otimes M(\tau)$ has a chain of submodules $E \otimes M(\tau) = M_1 \supset \cdots \supset M_m \supset 0$ such that $M_j/M_{j+1} \cong \bigoplus_{d_j} M(\tau+\mu_j)$. It is then clear that $\operatorname{pr}(E \otimes M(\tau)) = \operatorname{pr} M_1 \supset \cdots \supset \operatorname{pr} M_m \supset 0$ is a chain of submodules such that the subquotients $\operatorname{pr} M_j / \operatorname{pr} M_{j+1}$ are isomorphic to $\bigoplus_{d_j} M(\tau+\mu_j)$, if $\tau + \mu_j \in \mathcal{W} \cdot (\tau+\nu)$, otherwise they are 0. Since μ_j and ν are integral weights, we get $\mu_j = w \cdot (\tau+\nu) - \tau$ only for $w \in \mathcal{W}_\tau$. For $w = e$ this yields $\mu_{j_0} = \nu$, for $w = s_\alpha$ we obtain $\mu_{j_1} = \mu_0$ since $\tau + \mu_0 = s_\alpha \cdot (\tau+\nu)$. By assumption we have $\mu_{j_0} = \nu < \mu_0 = \mu_{j_1}$, and thus $j_0 < j_1$. Therefore, if we omit in this chain trivial submodules we obtain a chain $\operatorname{pr}\big(E \otimes M(\tau)\big) = \operatorname{pr} M_{j_0} =: R_1 \supset \operatorname{pr} M_{j_1} =: R_2 \supset 0$ such that $R_2 \cong \bigoplus_{d_{j_1}} M(\tau+\mu_0) = \bigoplus_{d_{j_1}} M(s_\alpha \cdot (\tau+\nu))$ and $R_1/R_2 \cong \bigoplus_{d_{j_0}} M(\tau+\nu) \cong M(\tau+\nu)$. By construction of the M_i this module is generated by $\operatorname{pr}_{\chi(\tau+\nu)}(e_\nu \otimes v_\tau)/R_2$. Now $\mu_0 > \nu$ forces $\langle \tau+\nu+\rho, \alpha^\vee \rangle < 0$ and together with $\mathcal{W}_{\tau+\nu} = \mathcal{W}_\tau = \langle s_\alpha \rangle$ this implies $M(\tau+\nu) = L(\tau+\nu)$ (Lemma 7). $\square$

For $\mu \in \mathfrak{h}^*$ define maps $H_\mu : \mathfrak{h}^* \times \mathfrak{h}^* \to k$ by $H_\mu(\lambda, \tau) := \langle \lambda - \tau - \mu, h_\mu \rangle$ $\forall \lambda, \tau \in \mathfrak{h}^*, h_\mu \in \mathfrak{h}$. For $\tau \in \mathfrak{h}^*$ set $p_\tau := \prod_{\mu > \nu, \mu \in P(E)} \operatorname{sym} H_\mu(-, \tau)$ and $z_\tau := \xi^{-1}(p_\tau) \in \mathfrak{Z}$ the preimage of p_τ under the Harish-Chandra isomorphism $\xi : \mathfrak{Z} \xrightarrow{\sim} \mathcal{P}(\mathfrak{h}^*)^{\mathcal{W}\cdot}$.

Lemma 9. *Let $\tau_0 \in \mathfrak{h}^*$ with $\mathcal{W}_{\tau_0} = \langle s_\alpha \rangle$ and $-\langle \rho, \alpha^\vee \rangle \leq \langle \tau_0, \alpha^\vee \rangle \leq -\langle \nu + \rho, \alpha^\vee \rangle - 1$. Assume the $h_\mu \in \mathfrak{h}$ to be chosen such that*

(a) *$\langle \alpha, h_{\mu_0} \rangle \neq 0$ for $\mu_0 := \nu - \langle \tau_0 + \nu + \rho, \alpha^\vee \rangle \alpha$ and*
(b) *$H_\mu\big(w \cdot (\tau_0 + \nu), \tau_0\big) \neq 0$ for $(w, \mu) \neq (s_\alpha, \mu_0), \forall w \in \mathcal{W}, \mu \in P(E)$ with $\mu > \nu$.*

Then $z_{\tau_0} \notin \operatorname{Ann}_{\mathfrak{Z}} P(\tau_0 + \nu)$.

Proof. We will first give an explicit description of the annihilator $\operatorname{Ann}_{\mathfrak{Z}} P(\tau_0 + \nu)$ by a theorem of Soergel [So]. For this let $\lambda \in \mathfrak{h}^*$, such that for all $\beta \in R^+ \cap R_\lambda$ we have $\langle \lambda + \rho, \beta^\vee \rangle \geq 0$. Denote by $w_\lambda \in \mathcal{W}_\lambda$ the longest element with respect to the Bruhat ordering. Then define for $\mu \in \mathfrak{h}^*$ the map

$$\begin{array}{rccc} \mu^+ : & \mathcal{P}(\mathfrak{h}^*) & \longrightarrow & \mathcal{P}(\mathfrak{h}^*) \\ & p & \mapsto & \mu^+(p) \end{array}$$

by $\big(\mu^+(p)\big)(\tau) := p(\tau + \mu)$ for all $\tau \in \mathfrak{h}^*$. Then [So, 2.2]:

$$\xi^{-1}(p) \in \operatorname{Ann}_{\mathfrak{Z}} P(w_\lambda \cdot \lambda) \Longleftrightarrow \lambda^+(p) \in \big(\mathcal{P}^+(\mathfrak{h}^*)\big)^{\mathcal{W}_\lambda} \mathcal{P}(\mathfrak{h}^*).$$

Here $\mathcal{P}^+(\mathfrak{h}^*)$ denotes the set of polynomial functions on $\mathfrak{h}^*$ without constant term. We want to describe the annihilator of $P(\tau_0+\nu)$ and choose for this $\lambda_0 := s_\alpha \cdot (\tau_0+\nu) = \tau_0+\mu_0 = \tau_0+\nu-\langle \tau_0+\nu+\rho, \alpha^\vee\rangle\alpha$. We then have $\mathcal{W}_{\lambda_0} = \mathcal{W}_{s_\alpha\cdot(\tau_0+\nu)} = \mathcal{W}_{\tau_0+\nu} = \mathcal{W}_{\tau_0} = \langle s_\alpha\rangle$ and hence $R_{\lambda_0} = \{\alpha, -\alpha\}$. By assumption we know that $\langle \lambda_0+\rho, \alpha^\vee\rangle \geq 0$ and obtain for all $\beta \in R^+\cap R_{\lambda_0}$ that $\langle \lambda_0+\rho, \beta^\vee\rangle \geq 0$. Now $w_{\lambda_0} = s_\alpha$ is the longest element in $\mathcal{W}_{\lambda_0}$ and we may apply Soergel's theorem with $\lambda := \lambda_0 = s_\alpha\cdot(\tau_0+\nu)$. Assume we had $\xi^{-1}(p) \in \operatorname{Ann}_{\mathfrak{Z}} P(w_{\lambda_0}\cdot\lambda_0) = \operatorname{Ann}_{\mathfrak{Z}} P(\tau_0+\nu)$ for $p = \sum_{i=1}^n p_i q_i$ with $p_i \in \mathcal{P}^+(\mathfrak{h}^*)\big)^{\mathcal{W}_{\lambda_0}}$ and $q_i \in \mathcal{P}(\mathfrak{h}^*)$. Then it follows for all p_i that

$$p_i(\lambda_0+\mu) = \left(\lambda_0^+(p_i)\right)(\mu) = \left(\lambda_0^+(p_i)\right)(s_\alpha\mu) = p_i(\lambda_0+s_\alpha\mu)\ \forall\mu\in\mathfrak{h}^*.$$

For $\mu=\alpha$ we obtain $p_i(\lambda_0+\alpha) = p_i(\lambda_0-\alpha)$ and this forces the derivative of p in direction of α to vanish at the point λ_0. Let's check this condition for p_τ. By definition we have for all $\lambda\in\mathfrak{h}^*$: $p_\tau(\lambda) = \prod_{\mu>\nu, w\in\mathcal{W}} H_\mu(w\cdot\lambda,\tau)$. If we define $\bar{p}_\tau$ by $\bar{p}_\tau H_{\mu_0}(-,\tau) = p_\tau$, we obtain

$$\bar{p}_\tau(\lambda) := \prod_{\substack{(w,\mu)\neq(e,\mu_0)\\ w\in\mathcal{W},\mu>\nu}} H_\mu(w\cdot\lambda,\tau).$$

Let p'_τ denote the derivative of p_τ in direction α. By the product rule we have $p'_\tau = \bar{p}'_\tau H_{\mu_0}(-,\tau) + \bar{p}_\tau H'_{\mu_0}(-,\tau)$.
Since $s_\alpha\cdot(\tau_0+\nu) = \tau_0+\mu_0$ it follows that $H_{\mu_0}\big(s_\alpha\cdot(\tau_0+\nu),\tau_0\big) = 0$ and we get $p'_{\tau_0}\big(s_\alpha\cdot(\tau_0+\nu)\big) = \bar{p}_{\tau_0}\big(s_\alpha\cdot(\tau_0+\nu)\big)H'_{\mu_0}\big(s_\alpha\cdot(\tau_0+\nu),\tau_0\big)$. But now neither of these two factors is zero because

$$\begin{aligned}\bar{p}_{\tau_0}\big(s_\alpha\cdot(\tau_0+\nu)\big) &= \prod_{\substack{(w,\mu)\neq(e,\mu_0)\\ w\in\mathcal{W},\mu>\nu}} H_\mu\big(w\cdot s_\alpha\cdot(\tau_0+\nu),\tau_0\big)\\ &= \prod_{\substack{(w,\mu)\neq(s_\alpha,\mu_0)\\ w\in\mathcal{W},\mu>\nu}} H_\mu\big(w\cdot(\tau_0+\nu),\tau_0\big)\end{aligned}$$

and by assumption (b) none of the factors $H_\mu\big(w\cdot(\tau_0+\nu),\tau_0\big)$ vanishes, hence $\bar{p}_{\tau_0}\big(s_\alpha\cdot(\tau_0+\nu)\big)\neq 0$. On the other hand, also $H'_{\mu_0}\big(s_\alpha\cdot(\tau_0+\nu),\tau_0\big)\neq 0$ since by assumption (a) we have: $H'_{\mu_0}(\lambda,\tau_0) = \frac{\mathrm{d}}{\mathrm{d}\,t}_{|t=0}\langle \lambda+t\alpha-\tau_0-\mu_0, h_{\mu_0}\rangle = \langle\alpha, h_{\mu_0}\rangle \neq 0$.
Together this implies $p'_{\tau_0}\big(s_\alpha\cdot(\tau_0+\nu)\big)\neq 0$ and therefore $\xi(p_{\tau_0}) = z_{\tau_0}$ cannot be contained in the annihilator of $P(\tau_0+\nu)$. □

4. The triangle function Δ

4.1. **Preliminaries.** Let M be a representation of $\mathfrak{g}$ and E a vector space. Then $E\otimes M$ is a representation of $\mathfrak{g}$ via $X(e\otimes m) := e\otimes Xm$ for all $X\in\mathfrak{g}, e\in E$ and $m\in M$. If in addition E is also a representation of $\mathfrak{g}$, then we obtain a second $\mathfrak{g}$-operation on $E\otimes M$ via $X(e\otimes m) := Xe\otimes m + e\otimes Xm$. To distinguish these two representations we denote the first one by $E\hat{\otimes}M$. Let now E be a finite dimensional representation of $\mathfrak{g}$ and $\nu\in P$ a weight of E.

Lemma 10. *Let $\tau \in \mathfrak{h}^*$ such that $\chi(\tau+\nu) \neq \chi(\tau+\mu)$ for all $\mu \in P(E)$ with $\mu \neq \nu$. Then there exists a unique natural isomorphism*

$$\mathrm{can}: \ E_\nu \hat{\otimes} M(\tau+\nu) \xrightarrow{\sim} \mathrm{pr}_{\chi(\tau+\nu)}(E \otimes M(\tau)),$$

such that $\mathrm{can}(e \hat{\otimes} v_{\tau+\nu}) = \mathrm{pr}_{\chi(\tau+\nu)}(e \otimes v_\tau)$ *for all $e \in E_\nu$.*

Remark. For a *generic* weight, i.e. a weight $\tau \in \mathfrak{h}^*$ such that $\langle \tau, \alpha^\vee \rangle \notin \mathbb{Z}$ for all $\alpha \in R$, the central characters $\chi(\tau+\mu)$ for $\mu \in P(E)$ are pairwise distinct. In particular, in this case the condition of the lemma is always satisfied.

Proof. By the so-called tensor identity we have a canonical isomorphism

$$\mathfrak{U} \otimes_{\mathfrak{U}(\mathfrak{b})} (E \otimes k_\tau) \xrightarrow{\sim} E \otimes (\mathfrak{U} \otimes_{\mathfrak{U}(\mathfrak{b})} k_\tau)$$

such that $u \otimes (e \otimes a) \mapsto u(e \otimes (1 \otimes a))$. Call the left hand side F, the right hand side is $E \otimes M(\tau)$. Denote by $\mu_1, \dots, \mu_m$ the weights of E. The filtration $E \otimes M(\tau) = M_1 \supset \dots \supset M_m \supset 0$, where the subquotients are isomorphic to direct sums of Verma modules (see page 4), induces a filtration of F : $\mathfrak{U} \otimes_{\mathfrak{U}(\mathfrak{b})} (E \otimes k_\tau) = F = F_1 \supset F_2 \supset \dots \supset F_m \supset 0$ such that

$$F_j / F_{j+1} \cong \mathfrak{U} \otimes_{\mathfrak{U}(\mathfrak{b})} (E_{\mu_j} \otimes k_\tau).$$

Now here the right hand side is canonically isomorphic to $E_{\mu_j} \hat{\otimes} M(\tau+\mu_j)$ (by mapping $e \otimes u v_{\tau+\mu_j} \mapsto u \otimes (e \otimes 1)$) and in particular, we have that $\chi(\tau+\mu_j)(F_j/F_{j+1}) = 0$.
Let now $\nu = \mu_i$ for a fixed i. By the condition on τ we know that $\chi(\tau+\nu) \neq \chi(\tau+\mu_j)$ for all $j \neq i$, $j \in \{1, \dots, m\}$ and hence $\mathrm{pr}_{\chi(\tau+\nu)}(F_j/F_{j+1}) = 0$ for all $j \neq i$. Thus we get $\mathrm{pr}_{\chi(\tau+\nu)} F = \mathrm{pr}_{\chi(\tau+\nu)} F_1 = \ldots = \mathrm{pr}_{\chi(\tau+\nu)} F_i$ and also $\mathrm{pr}_{\chi(\tau+\nu)} F_{i+1} = \ldots = \mathrm{pr}_{\chi(\tau+\nu)} F_m = 0$. We conclude that $\mathrm{pr}_{\chi(\tau+\nu)} F \subset F_i$ and that $F_{i+1} \subset \ker(\mathrm{pr}_{\chi(\tau+\nu)} : F_i \twoheadrightarrow \mathrm{pr}_{\chi(\tau+\nu)} F)$. Since $\mathrm{pr}_{\chi(\tau+\nu)}(F_i/F_{i+1}) = F_i/F_{i+1}$, we know even that F_{i+1} is equal to this kernel. This now induces a natural isomorphism

$$F_i/F_{i+1} \xrightarrow{\sim} \mathrm{pr}_{\chi(\tau+\nu)} F$$

such that

$$\begin{array}{ccc} E_\nu \hat{\otimes} M(\tau+\nu) & \xrightarrow{\sim} & \mathrm{pr}_{\chi(\tau+\nu)} F \\ e \hat{\otimes} (u v_{\tau+\nu}) & \longmapsto & \mathrm{pr}_{\chi(\tau+\nu)}(u \otimes (e \otimes 1)) \end{array}$$

We apply the tensor identity $F \cong E \otimes M(\tau)$ and the lemma follows. $\square$

Let us recall the theorem of Bernstein and Gelfand [BG] for projective functors. For this denote by $\mathcal{M}$ the category of all $\mathfrak{Z}$-finite $\mathfrak{g}$-modules and for $\chi \in \mathrm{Max}\mathfrak{Z}$ let

$$\begin{aligned} \mathcal{M}(\chi) &:= \{M \in \mathcal{M} \mid \chi M = 0\} \\ \mathcal{M}^\infty(\chi) &:= \{M \in \mathcal{M} \mid \text{ for all } m \in M \text{ exists } n \in \mathbb{N} \text{ such that } \chi^n m = 0\}. \end{aligned}$$

A *projective χ-functor* is then a functor $F : \ \mathcal{M}(\chi) \to \mathcal{M}$, which is isomorphic to a direct summand of a functor $E \otimes$ for a finite dimensional representation E. In particular, the restriction of the translation functor $T_\tau^{\tau+\nu} : \ \mathcal{M}^\infty(\chi(\tau)) \to \mathcal{M}^\infty(\chi(\tau+\nu))$ to the subcategory $\mathcal{M}(\chi(\tau))$ is a projective $\chi(\tau)$-functor. For two projective χ-functors $F, \tilde{F} : \ \mathcal{M}(\chi) \to \mathcal{M}$

denote by $\mathrm{Hom}_{\mathcal{M}(\chi)\to}(F,\tilde{F})$ the space of all natural transformations from F to $\tilde{F}$.

Theorem. [BG, 3.5] *Let $F,\tilde{F}:\ \mathcal{M}(\chi)\to\mathcal{M}$ be projective χ-functors and let $\tau\in\mathfrak{h}^*$ such that $\chi(\tau)=\chi$ and $M(\tau)$ is projective. Then the obvious map*

$$\mathrm{Hom}_{\mathcal{M}(\chi(\tau))\to}(F,\tilde{F})\xrightarrow{\sim}\mathrm{Hom}_{\mathfrak{g}}(FM(\tau),\tilde{F}M(\tau))$$

is an isomorphism.

Remark. (i) The Verma module $M(\tau)$ is projective if and only if $\langle\tau+\rho,\alpha^\vee\rangle\notin\{-1,-2,\dots\}$ for all $\alpha\in R^+$. In particular, for a generic weight τ the Verma module $M(\tau)$ is always projective.

(ii) Note that in [BG] the Verma module with highest weight τ is denoted by $M_{\tau+\rho}$. Accordingly, the theorem there is formulated for all τ with $\langle\tau,\alpha^\vee\rangle\notin\{-1,-2,\dots\}$ for all $\alpha\in R^+$.

4.2. **Definition of Δ.** For $\nu\in P$ let $E(\nu)$ be the finite dimensional irreducible $\mathfrak{g}$-module with extremal weight ν and let $x\in\mathcal{W}$. For a weight $\tau\in\mathfrak{h}^*$ with $\chi(\tau+\nu)\neq\chi(\tau+\mu)$ for all $\mu\in P(E)$ with $\mu\neq\nu$, Lemma 10 yields a canonical isomorphism

$$E(\nu)_{x\nu}\hat{\otimes}M(x\cdot(\tau+\nu))\xrightarrow{\sim}\mathrm{pr}_{\chi(\tau+\nu)}(E(\nu)\otimes M(x\cdot\tau))=T_\tau^{\tau+\nu}M(x\cdot\tau).$$

Let now $\nu,\mu\in P$ be integral weights. We set $E':=E(\nu)$, $E'':=E(\mu)$ and $E:=E(\nu+\mu)$. For generic τ consider the following sequence of isomorphisms:

$$\begin{aligned}&\mathrm{Hom}_{\mathcal{M}(\chi(\tau))\to}\left(T_{\tau+\nu}^{\tau+\nu+\mu}\circ T_\tau^{\tau+\nu},T_\tau^{\tau+\nu+\mu}\right)\\ &\xrightarrow{\sim}\mathrm{Hom}_{\mathfrak{g}}\left(T_{\tau+\nu}^{\tau+\nu+\mu}T_\tau^{\tau+\nu}M(x\cdot\tau),T_\tau^{\tau+\nu+\mu}M(x\cdot\tau)\right)\\ &\xrightarrow{\sim}\mathrm{Hom}_{\mathfrak{g}}\left(E''_{x\mu}\hat{\otimes}E'_{x\nu}\hat{\otimes}M(x\cdot(\tau+\nu+\mu)),E_{x(\mu+\nu)}\hat{\otimes}M(x\cdot(\tau+\nu+\mu))\right)\\ &\xrightarrow{\sim}\mathrm{Hom}_k(E''_{x\mu}\otimes E'_{x\nu},E_{x(\mu+\nu)})\\ &\xrightarrow{\sim}E''^*_{x\mu}\otimes E'^*_{x\nu}\otimes E_{x(\mu+\nu)}\end{aligned}$$

Here, we obtain the first isomorphism by the Theorem of Bernstein-Gelfand, the second is due to Lemma 10, the others are obvious. We call this map $\mathrm{nat}(\mu,\nu;x)(\tau)$ and define for generic τ and for the triangle

$$\begin{tikzcd} & \cdot \arrow[dr, "\mu"] & \\ \tau\,\cdot \arrow[ur, "\nu"] \arrow[rr, "\nu+\mu"] & & \cdot \end{tikzcd}$$

the value of the triangle function Δ by

$$\Delta(\mu,\nu;x)(\tau):=\det\left(x^{-1}\circ\mathrm{nat}(\mu,\nu;x)(\tau)\circ(\mathrm{nat}(\mu,\nu;e)(\tau))^{-1}\right).$$

We have yet to explain the map

$$x^{-1}:E(\mu)^*_{x\mu}\otimes E(\nu)^*_{x\nu}\otimes E(\mu+\nu)_{x(\nu+\mu)}\to E(\mu)^*_\mu\otimes E(\nu)^*_\nu\otimes E(\mu+\nu)_{\nu+\mu}.$$

For this let G be a simply connected algebraic group with Lie algebra $\mathfrak{g}$ and $T\subset G$ a maximal torus with Lie algebra $\mathfrak{h}$. Each finite dimensional representation E of $\mathfrak{g}$ is in a natural way a representation of G. The operation of $N_G(T)$, the normalizer of T in G, on E stabilizes E_0 and factors over an

operation of $\mathcal{W} = N_G(T)/T$. The map x^{-1} is given by this operation of $\mathcal{W}$ on the zero weight space $(E(\mu)^* \otimes E(\nu)^* \otimes E(\nu+\mu))_0$.
Now take for τ not only any, but rather *the* generic weight: For this denote by $S := S_k(\mathfrak{h})$ the symmetric algebra of $\mathfrak{h}$ and let $K := \mathrm{Quot}(S)$ be its quotient field. We then have a k-linear map $\mathfrak{h} \hookrightarrow S \hookrightarrow K$ and obtain thus a K-linear map $\tau : K \otimes_k \mathfrak{h} \longrightarrow K$ such that the following diagram commutes

$$\begin{array}{ccc} \mathfrak{h} & \longrightarrow & S \\ \big\downarrow & & \big\downarrow \\ K \otimes_k \mathfrak{h} & \xrightarrow{\tau} & K \end{array}$$

With this τ (= tautologous) we then obtain $\Delta(\mu, \nu; x)(\tau) \in K^\times$. These are the triangle functions.

4.3. **Uncanonical definition of Δ.** We defined the triangle functions by a series of canonical isomorphisms. For our purposes it is sometimes more convenient to realize the triangle functions in the following – uncanonical – way: Let $\nu \in P$, $x \in \mathcal{W}$ and choose a fixed extremal weight vector $0 \neq e_\nu \in E(\nu)_\nu$. Since extremal weight spaces are one dimensional, this choice is unique up to non-zero scalar. By Lemma 10 we obtain for generic $\tau \in \mathfrak{h}^*$ a – no longer canonical – isomorphism

$$\begin{array}{rccc} F_{x\nu}(x \cdot \tau): & M(x \cdot (\tau+\nu)) & \xrightarrow{\sim} & T_\tau^{\tau+\nu} M(x \cdot \tau) \\ & v_{x\cdot(\tau+\nu)} & \mapsto & \mathrm{pr}_{\chi(\tau+\nu)}(\dot{x} e_\nu \otimes v_{x\cdot\tau}) \end{array}$$

Here $\dot{x} \in G$ denotes a pre-image of $x \in \mathcal{W} \cong N_G T/T$. We have $\dot{x} e_\nu \in E(\nu)_{x\nu}$. Let now $\mu \in P$ be another weight. Choose $\tilde{e}_\mu \in E(\mu)_\mu$ and $\bar{e}_{\mu+\nu} \in E(\mu+\nu)_{\mu+\nu}$, both non-zero, and consider for generic τ the following sequence of isomorphisms:

$$\begin{aligned} &\mathrm{Hom}_{\mathcal{M}(\chi(\tau))\to}\left(T_{\tau+\nu}^{\tau+\nu+\mu} \circ T_\tau^{\tau+\nu}, T_\tau^{\tau+\nu+\mu}\right) \\ &\qquad \xrightarrow{\sim} \mathrm{Hom}_{\mathfrak{g}}\left(T_{\tau+\nu}^{\tau+\nu+\mu} T_\tau^{\tau+\nu} M(x\cdot\tau), T_\tau^{\tau+\nu+\mu} M(x\cdot\tau)\right) \\ &\qquad \xrightarrow{\sim} \mathrm{Hom}_{\mathfrak{g}}\left(M(x\cdot(\tau+\nu+\mu)), M(x\cdot(\tau+\nu+\mu))\right) \\ &\qquad \xrightarrow{\sim} k \end{aligned}$$

Denote this map by $\mathrm{Nat}(\mu, \nu; x)(\tau)$. Here again the first isomorphism is clear by the Theorem of Bernstein-Gelfand, the second is due to the the maps $F_{x(\nu+\mu)}(x\cdot\tau)$ and $F_{x\mu}(x\cdot(\tau+\mu)) \circ T_{\tau+\nu}^{\tau+\nu+\mu} F_{x\nu}(x\cdot\tau)$. We then obtain

$$\Delta(\mu,\nu;x)(\tau) = \mathrm{Nat}(\mu,\nu;x)(\tau) \circ (\mathrm{Nat}(\mu,\nu;e)(\tau))^{-1}(1).$$

It follows that this characterization of Δ is independent of the choice of the weight vectors $e_\nu, \tilde{e}_\mu$ and $\bar{e}_{\mu+\nu}$, and also independent of the choice of the pre-image $\dot{x}$ of x.

5. Bernstein's relative trace and a special case of Δ

5.1. **The relative trace.** Let us recall the definition of the *relative trace* tr_E defined by Bernstein in [Be]. Let $\mathfrak{g}$-mod be the category of all $\mathfrak{g}$-modules and let E be a finite dimensional $\mathfrak{g}$-module. Denote by F_E : $\mathfrak{g}$-mod $\to$ $\mathfrak{g}$-mod the functor defined by $F_E(M) := E \otimes M$. The relative trace $\mathrm{tr}_E : \mathrm{End}_{\mathfrak{g}\text{-mod}}(F_E) \to \mathrm{End}_{\mathfrak{g}\text{-mod}}(\mathrm{Id})$ is then a morphism from the

endomorphisms of the functor F_E to the endomorphisms of the identity functor on $\mathfrak{g}$-mod, defined by

$$\begin{array}{rccc} \mathrm{tr}_E^M : & \mathrm{End}_{\mathfrak{g}}(E \otimes M) & \to & \mathrm{End}_{\mathfrak{g}} M \\ & a & \mapsto & \mathrm{tr}_E^M(a) \end{array}$$

where

$$\mathrm{tr}_E^M(a) : \ M \xrightarrow{\ \mathrm{i}\ } E^* \otimes E \otimes M \xrightarrow{\mathrm{id} \otimes a} E^* \otimes E \otimes M \xrightarrow{\mathrm{cont} \otimes \mathrm{id}} M \cdot$$

Here, i is the map $M \overset{\mathrm{j}}{\hookrightarrow} \mathrm{End}_{\mathfrak{g}}(E) \otimes M \xrightarrow{\mathrm{c}} E^* \otimes E \otimes M$ with $\mathrm{j}(m) := \mathrm{id}_E \otimes m$ and c the canonical isomorphism $\mathrm{End}_{\mathfrak{g}}(E) \cong E^* \otimes E$. The map cont : $E^* \otimes E \to k$ denotes the evaluation map.
Bernstein has calculated an explicit formula for the relative trace, by considering it as a map from $\mathcal{P}(\mathfrak{h}^*)^{\mathcal{W}\cdot}$ to itself in the following way: First, we identify $\mathrm{End}_{\mathfrak{g}\text{-mod}}(\mathrm{Id}) \cong \mathfrak{Z}$ with the center of the enveloping algebra $\mathfrak{U}$, then we make use of the natural morphism $\mathfrak{Z} \to \mathrm{End}_{\mathfrak{g}\text{-mod}}(F_E)$ and composing this with the trace map we obtain $\mathrm{tr}_E : \mathfrak{Z} \to \mathfrak{Z}$. By means of the Harish-Chandra isomorphism $\xi : \mathfrak{Z} \xrightarrow{\sim} \mathcal{P}(\mathfrak{h}^*)^{\mathcal{W}\cdot}$ (normalized by $z - \xi(z) \in \mathfrak{U}\mathfrak{n}$) we may then regard tr_E as an endomorphism of $\mathcal{P}(\mathfrak{h}^*)^{\mathcal{W}\cdot}$.
Define now on $\mathcal{P}(\mathfrak{h}^*)$ a convolution $f \mapsto P(E) * f$ by $(P(E) * f)(\lambda) := \sum_{\mu \in P(E)} f(\lambda + \mu)$, where the sum is taken over all weights $\mu \in P(E)$ with their multiplicities. Set $\Lambda(\lambda) := \prod_{\alpha \in R^+} \langle \lambda + \rho, \alpha^\vee \rangle$. Then we have

Theorem. [Be] *$\mathrm{tr}_E(f) = \Lambda^{-1}(P(E) * \Lambda f)$ for all $f \in \mathcal{P}(\mathfrak{h}^*)^{\mathcal{W}\cdot}$.*

If we choose now for M the Verma module $M(\lambda)$ we can associate to each endomorphism $f \in \mathrm{End}_{\mathfrak{g}}(E \otimes M(\lambda))$ an element $\mathrm{tr}_E^{M(\lambda)}(f)$ of $\mathrm{End}_{\mathfrak{g}}(M(\lambda)) \cong k$. As endomorphism of $M(\lambda)$ this element operates on $M(\lambda)$ by multiplication with the scalar $(\mathrm{tr}_E^{M(\lambda)}(f))(\lambda)$ and we obtain by Bernstein's Theorem for all $\lambda \in \mathfrak{h}^*$ and for all $w \in \mathcal{W}$

$$\begin{aligned} \big(\mathrm{tr}_E^{M(w\cdot\lambda)}(f)\big)(\lambda) &= \big(\Lambda^{-1}(P(E) * \Lambda f)\big)(\lambda) \\ &= (\Lambda(\lambda))^{-1} \sum_{\mu \in P(E)} \Lambda(\lambda + \mu) f(\lambda + \mu). \end{aligned}$$

5.2. **The special case $\Delta(-\nu, \nu; w_0)$.** Let now $E = E(\nu)$, $w_0 \in \mathcal{W}$ the longest element and $\mathrm{pr}_{\chi(\tau+\nu)} \in \mathrm{End}_{\mathfrak{g}}(E(\nu) \otimes M(w_0 \cdot \tau))$ the projection on the central character $\chi(\tau + \nu)$. Then $\mathrm{tr}_{E(\nu)}^{M(w_0\cdot\tau)}(\mathrm{pr}_{\chi(\tau+\nu)})$ is an element in $\mathrm{End}_{\mathfrak{g}}(M(w_0 \cdot \tau)) \cong k$ and we have

Theorem 2. *Let $\nu \in P^+$ be a dominant weight and $\tau \in \mathfrak{h}^*$ generic. Then*

$$\Delta(-\nu, \nu; w_0)(\tau) = \mathrm{tr}_{E(\nu)}^{M(w_0\cdot\tau)}(\mathrm{pr}_{\chi(\tau+\nu)}).$$

Proof. Postponed to 5.3. ☐

Regarding $\Delta(-\nu, \nu; w_0)$ as a polynomial function on $\mathfrak{h}^*$ we obtain for this special case an explicit formula:

Corollary. *Let $\nu \in P^+$ be a dominant weight and $\tau \in \mathfrak{h}^*$ generic. Then*

$$\Delta(-\nu, \nu; w_0)(\tau) = \prod_{\alpha \in R^+} \frac{\langle \tau + \nu + \rho, \alpha^\vee \rangle}{\langle \tau + \rho, \alpha^\vee \rangle} \ .$$

Proof. Under the morphism $\mathcal{P}(\mathfrak{h}^*)^{\mathcal{W}\cdot} \cong \mathfrak{Z} \to \mathrm{End}_{\mathfrak{g}\text{-mod}}(F_E)$ the projection $\mathrm{pr}_{\chi(\tau+\nu)}$ is the image of a polynomial function, which takes value 1 at all weights $\lambda \in \mathcal{W}\cdot(\tau+\nu)$ and vanishes at all other weights $\tau+\mu$ with $\mu \in P(E)$. Call this polynomial function $\bar{\mathrm{pr}}$. By Bernstein's formula for the relative trace we then obtain

$$\big(\operatorname{tr}_{E(\nu)}^{M(w_0\cdot\tau)}(\bar{\mathrm{pr}})\big)(\tau) = (\Lambda(\tau))^{-1} \sum_{\mu\in P(E(\nu))} \Lambda(\tau+\mu)\,\bar{\mathrm{pr}}(\tau+\mu)$$

and by definition of $\bar{\mathrm{pr}}$ the value of $\bar{\mathrm{pr}}(\tau+\mu)$ for $\mu \in P(E)$ does not vanish if and only if there is a $w \in \mathcal{W}$ such that $w\cdot(\tau+\mu) = (\tau+\nu)$. Since τ is generic, this is only possible for $w = e$ and hence $\mu = \nu$. In this case we have $\bar{\mathrm{pr}}(\tau+\nu) = 1$ and since furthermore $\dim E(\nu)_\nu = 1$, we get $\sum_{\mu\in P(E(\nu))} \Lambda(\tau+\mu)\,\bar{\mathrm{pr}}(\tau+\mu) = \Lambda(\tau+\nu)$. The claim now follows by Theorem 2 and the equation

$$\big(\operatorname{tr}_{E(\nu)}^{M(w_0\cdot\tau)}(\bar{\mathrm{pr}})\big)(\tau) = \frac{\Lambda(\tau+\nu)}{\Lambda(\tau)} = \prod_{\alpha\in R^+} \frac{\langle \tau+\nu+\rho, \alpha^\vee\rangle}{\langle \tau+\rho, \alpha^\vee\rangle}\,.$$

□

5.3. **Proof of Theorem 2.** First we make some more general preliminary remarks.

5.3.1. *The adjunctions* $(T_\tau^{\tau+\nu}, T_{\tau+\nu}^\tau)$ *and* $(T_{\tau+\nu}^\tau, T_\tau^{\tau+\nu})$. Let $\mathcal{A}$ and $\mathcal{B}$ be categories and $F : \mathcal{A} \to \mathcal{B}$, $G : \mathcal{B} \to \mathcal{A}$ two functors. Then an adjunction (F,G) of F and G is a family of isomorphisms

$$(F,G)_{M,N} := (F,G) : \mathrm{Hom}_{\mathcal{B}}(FM,N) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{A}}(M,GN),$$

which is natural in M and N ($M \in \mathcal{A},\ N \in \mathcal{B}$).
For example, for E a finite dimensional $\mathfrak{g}$-module we obtain an adjunction (F_E, F_{E^*}) of F_E: $\mathfrak{g}$-mod $\to$ $\mathfrak{g}$-mod and F_{E^*}: $\mathfrak{g}$-mod $\to$ $\mathfrak{g}$-mod as the composition

$$\mathrm{Hom}_{\mathfrak{g}}(E\otimes M, N) \to \mathrm{Hom}_{\mathfrak{g}}(E^*\otimes E\otimes M, E^*\otimes N) \to \mathrm{Hom}_{\mathfrak{g}}(M, E^*\otimes N).$$

Here, the first map is given by $f \mapsto \mathrm{id}_{E^*}\otimes f$, the second by $g \mapsto g\circ \mathrm{i}$. Interchanging E and E^* we obtain in the same way an adjunction (F_{E^*}, F_E). Its inverse is the composition

$$\mathrm{Hom}_{\mathfrak{g}}(M, E\otimes N) \to \mathrm{Hom}_{\mathfrak{g}}(E^*\otimes M, E^*\otimes E\otimes N) \to \mathrm{Hom}_{\mathfrak{g}}(E^*\otimes M, N),$$

where the first map is again given by $f \mapsto \mathrm{id}_{E^*}\otimes f$ and the second by $g \mapsto (\mathrm{cont}\otimes \mathrm{id}_N)\circ g$. Let now $\nu \in P$ be an integral weight. Each identification $\varphi : E(\nu)^* \xrightarrow{\sim} E(-\nu)$ then defines adjunctions $(F_{E(\nu)}, F_{E(-\nu)})$ and $(F_{E(-\nu)}, F_{E(\nu)})$. More precisely, we have

$$\begin{array}{rccc} (F_{E(\nu)}, F_{E(-\nu)}): & \mathrm{Hom}_{\mathfrak{g}}(E(\nu)\otimes M, N) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathfrak{g}}(M, E(-\nu)\otimes N) \\ & f & \mapsto & (\varphi\otimes f)\circ \mathrm{i} \end{array}$$

and

$$\begin{array}{rccc} (F_{E(-\nu)}, F_{E(\nu)})^{-1}: & \mathrm{Hom}_{\mathfrak{g}}(M, E(\nu)\otimes N) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathfrak{g}}(E(-\nu)\otimes M, N) \\ & g & \mapsto & (\mathrm{cont}\otimes \mathrm{id}_N)\circ(\varphi^{-1}\otimes g) \end{array}$$

Let now $\mathrm{i}_\chi : \mathcal{M}^\infty(\chi) \hookrightarrow \mathcal{M}$ denote the embedding functor. We then have in a natural way adjunctions $(\mathrm{i}_\chi, \mathrm{pr}_\chi)$ and $(\mathrm{pr}_\chi, \mathrm{i}_\chi)$. Since for the translation functor $T_\tau^{\tau+\nu} = \mathrm{pr}_{\chi(\tau+\nu)} \circ F_{E(\nu)} \circ \mathrm{i}_{\chi(\tau)}$, we thus obtain also adjunctions $(T_\tau^{\tau+\nu}, T_{\tau+\nu}^\tau)$ and $(T_{\tau+\nu}^\tau, T_\tau^{\tau+\nu})$.

5.3.2. *The natural transformations* adj^1 *and* adj^2. Let $M \in \mathcal{M}^\infty(\chi(\tau))$ and consider $\mathrm{Id} \in \mathrm{Hom}_{\mathfrak{g}}(T_\tau^{\tau+\nu}M, T_\tau^{\tau+\nu}M)$. By means of the two adjunctions $(T_\tau^{\tau+\nu}, T_{\tau+\nu}^\tau)$ and $(T_{\tau+\nu}^\tau, T_\tau^{\tau+\nu})^{-1}$ we get two maps as the images of Id:

$$\mathrm{adj}^1_M \in \mathrm{Hom}_{\mathfrak{g}}(M, T_{\tau+\nu}^\tau T_\tau^{\tau+\nu}M) \text{ and } \mathrm{adj}^2_M \in \mathrm{Hom}_{\mathfrak{g}}(T_{\tau+\nu}^\tau T_\tau^{\tau+\nu}M, M)$$

and one checks that we obtain in this way natural transformations $\mathrm{adj}^1 \in \mathrm{Hom}_{\mathcal{M}^\infty(\chi(\tau))\to}(\mathrm{Id}, T_{\tau+\nu}^\tau \circ T_\tau^{\tau+\nu})$ and $\mathrm{adj}^2 \in \mathrm{Hom}_{\mathcal{M}^\infty(\chi(\tau))\to}(T_{\tau+\nu}^\tau \circ T_\tau^{\tau+\nu}, \mathrm{Id})$. By composing these two maps, we get a canonical endomorphism of M:

$$M \xrightarrow{\mathrm{adj}^1_M} T_{\tau+\nu}^\tau T_\tau^{\tau+\nu}M \xrightarrow{\mathrm{adj}^2_M} M.$$

We want to describe this endomorphism in more detail. By definition, the adjunction $(T_\tau^{\tau+\nu}, T_{\tau+\nu}^\tau)$ is just the composition of adjunctions $(\mathrm{i}_{\chi(\tau)}, \mathrm{pr}_{\chi(\tau)}) \circ (F_{E(\nu)}, F_{E(-\nu)}) \circ (\mathrm{pr}_{\chi(\tau+\nu)}, \mathrm{i}_{\chi(\tau+\nu)})$ and one checks easily that the image of $\mathrm{Id} \in \mathrm{Hom}_{\mathfrak{g}}(T_\tau^{\tau+\nu}M, T_\tau^{\tau+\nu}M)$ under $(\mathrm{pr}_{\chi(\tau+\nu)}, \mathrm{i}_{\chi(\tau+\nu)})$ is precisely the projection $\mathrm{pr}_{\chi(\tau+\nu)}$. Thus, the image of Id under the adjunction $(T_\tau^{\tau+\nu}, T_{\tau+\nu}^\tau)$ can be described by the composition

$$\mathrm{adj}^1_M : M \xrightarrow{\ \mathrm{i}\ } E^* \otimes E \otimes M \xrightarrow{\varphi\otimes\mathrm{pr}} T_{\tau+\nu}^\tau T_\tau^{\tau+\nu}M.$$

Here, we put $E = E(\nu)$ and $\mathrm{pr} = \mathrm{pr}_{\chi(\tau+\nu)}$. Analogously, we obtain for adj^2_M the composition

$$\mathrm{adj}^2_M : T_{\tau+\nu}^\tau T_\tau^{\tau+\nu}M \xrightarrow{\varphi^{-1}\otimes\mathrm{pr}} E^* \otimes E \otimes M \xrightarrow{\mathrm{cont}\otimes\mathrm{id}_M} M$$

and taken together we have the following commutative diagram:

$$\begin{array}{ccccc}
M & \xrightarrow{\mathrm{adj}^1_M} & T_{\tau+\nu}^\tau T_\tau^{\tau+\nu}M & \xrightarrow{\mathrm{adj}^2_M} & M \\
{\scriptstyle \mathrm{i}}\big\downarrow & & {\scriptstyle \varphi\otimes\mathrm{id}_{E\otimes M}}\big\uparrow & & {\scriptstyle \mathrm{cont}\otimes\mathrm{id}_M}\big\uparrow \\
E^* \otimes E \otimes M & \xrightarrow{\mathrm{id}_{E^*}\otimes\mathrm{pr}} & E^* \otimes E \otimes M & \xrightarrow{\mathrm{id}_{E^*}\otimes\mathrm{pr}} & E^* \otimes E \otimes M
\end{array}$$

We thus obtain for $\mathrm{adj}^2_M \circ \mathrm{adj}^1_M$

$$M \xrightarrow{\ \mathrm{i}\ } E^* \otimes E \otimes M \xrightarrow{\mathrm{id}_{E^*}\otimes\mathrm{pr}_{\chi(\tau+\nu)}} E^* \otimes E \otimes M \xrightarrow{\mathrm{cont}\otimes\mathrm{id}_M} M.$$

Comparing this with the relative trace tr_E for $E = E(\nu)$, it follows immediately for all $M \in \mathcal{M}^\infty(\chi(\tau))$ that

$$\mathrm{adj}^2_M \circ \mathrm{adj}^1_M = \mathrm{tr}^M_{E(\nu)}(\mathrm{pr}_{\chi(\tau+\nu)}) \in \mathrm{End}_{\mathfrak{g}}(M)$$

or, regarded as natural transformation of the identity functor $\mathrm{Id} : \mathcal{M}^\infty(\chi(\tau)) \to \mathcal{M}^\infty(\chi(\tau))$ to itself

$$\mathrm{adj}^2 \circ \mathrm{adj}^1 = \mathrm{tr}_{E(\nu)}(\mathrm{pr}_{\chi(\tau+\nu)}).$$

Let now $\iota_\chi : \mathcal{M}(\chi) \hookrightarrow \mathcal{M}$ denote the embedding functor, let $F, G : \mathcal{M}^\infty(\chi) \to \mathcal{M}$ be functors and denote by $F(\chi)$, resp. $G(\chi)$ its restrictions to the subcategory $\mathcal{M}(\chi)$. Each natural transformation n from F to G can be regarded as natural transformation from $F(\chi)$ to $G(\chi)$ by first applying ι_χ

and then n. In particular, we obtain in this way the two natural transformations $\mathrm{adj}^1 \in \mathrm{Hom}_{\mathcal{M}(\chi(\tau))\to}\left(\mathrm{Id}(\chi), T^\tau_{\tau+\nu} \circ T^{\tau+\nu}_\tau(\chi)\right)$ and similarly $\mathrm{adj}^2 \in \mathrm{Hom}_{\mathcal{M}(\chi(\tau))\to}\left(T^\tau_{\tau+\nu} \circ T^{\tau+\nu}_\tau(\chi), \mathrm{Id}(\chi)\right)$.

5.3.3. *We have* $\Delta(-\nu,\nu;w_0)(\tau) = \mathrm{adj}^2_{M(w_0\cdot\tau)} \circ \mathrm{adj}^1_{M(w_0\cdot\tau)}$. By choosing for M the Verma module $M(\tau)$, we can assign to each $\tau \in \mathfrak{h}^*$ a canonical element $\mathrm{adj}^2_{M(\tau)} \circ \mathrm{adj}^1_{M(\tau)} \in \mathrm{End}(M(\tau)) \cong k$. We will see in the following, that for dominant ν this is precisely the triangle function $\Delta(-\nu,\nu;w_0)(\tau)$. For $x \in \mathcal{W}$ and τ generic we call $\phi_x(x\cdot\tau)$ the isomorphism $T^\tau_{\tau+\nu}T^{\tau+\nu}_\tau M(x\cdot\tau) \xrightarrow{\sim} M(x\cdot\tau)$ given by $(\phi_x(x\cdot\tau))^{-1} = T^\tau_{\tau+\nu}F_{x\nu}(x\cdot\tau) \circ F_{-x\nu}(x\cdot(\tau+\nu))$ (see 4.3). First we show that for dominant ν and generic τ the following diagram commutes:

$$\begin{tikzcd}
T^\tau_{\tau+\nu}T^{\tau+\nu}_\tau M(\tau) \arrow[rr, "\mathrm{adj}^2_{M(\tau)}"] \arrow[d, "\phi_e(\tau)"'] & & M(\tau) \\
M(\tau) \arrow[urr, "\mathrm{id}"'] & &
\end{tikzcd}$$

Let again denote $e_\nu \in E(\nu)_\nu$ the fixed extremal weight vector for a dominant integral weight ν. Since w_0 is the longest element in the Weyl group the dominance of ν implies that $e_{-\nu} := \dot{w}_0 e_\nu$ is a weight vector of weight $-\nu$. Here, $\dot{w}_0 \in G$ is a representative of $w_0 \in N_G(T)/T$.
Define now a pairing $E(-\nu) \times E(\nu) \to k$ by $\langle e_{-\nu}, e_\nu\rangle := 1$ and obtain thus an identification $\varphi : E(\nu)^* \xrightarrow{\sim} E(-\nu)$. Let $v_\tau \in M(\tau)$ be the canonical generator. To see that the above diagram commutes, it suffices to show that the pre-image of v_τ in $T^\tau_{\tau+\nu}T^{\tau+\nu}_\tau M(\tau)$ is mapped again to v_τ when applying $\mathrm{adj}^2_{M(\tau)}$. We have

$$\begin{aligned}
(\phi_e(\tau))^{-1}(v_\tau) &= \mathrm{pr}_{\chi(\tau)}\left(e_{-\nu} \otimes (\mathrm{pr}_{\chi(\tau+\nu)}(e_\nu \otimes v_\tau))\right) \\
&= \mathrm{pr}_{\chi(\tau)}(e_{-\nu} \otimes e_\nu \otimes v_\tau) \\
&= e_{-\nu} \otimes e_\nu \otimes v_\tau - \bigoplus_{\chi\neq\chi(\tau)} \mathrm{pr}_\chi(e_{-\nu} \otimes e_\nu \otimes v_\tau).
\end{aligned}$$

The first equation holds by definition of $\phi_e(\tau)$, the second follows since for dominant ν we have $\mathrm{pr}_{\chi(\tau+\nu)}(e_\nu \otimes v_\tau) = e_\nu \otimes v_\tau$ (see the example on page 6) and the third equation is just direct sum decomposition. Since adj^2_M was the composition

$$T^\tau_{\tau+\nu}T^{\tau+\nu}_\tau M \xrightarrow{\varphi^{-1}\otimes\mathrm{pr}} E^* \otimes E \otimes M \xrightarrow{\mathrm{cont}\otimes\mathrm{id}_M} M$$

we know in particular that $\mathrm{adj}^2_{M(\tau)}$ is a $\mathfrak{g}$-module homomorphism to $M(\tau)$. Then it is clear, that $\mathrm{adj}^2_{M(\tau)}$ maps $\bigoplus_{\chi\neq\chi(\tau)} \mathrm{pr}_\chi(e_{-\nu}\otimes e_\nu \otimes v_\tau)$ to zero since this element has wrong central character. For $e_{-\nu}\otimes e_\nu\otimes v_\tau$ we use the fact that $\langle e_{-\nu}, e_\nu\rangle = 1$ and obtain as image under $\mathrm{adj}^2_{M(\tau)}$ precisely v_τ. Taken together, we get $\mathrm{adj}^2_{M(\tau)} = \mathrm{id} \circ \phi_e(\tau)$, i.e. the above diagram commutes.
This now means that under the map $\mathrm{Nat}(-\nu,\nu;e)(\tau)$ the image of $\mathrm{adj}^2 \in \mathrm{Hom}_{\mathcal{M}(\chi(\tau))\to}\left(T^\tau_{\tau+\nu}\circ T^{\tau+\nu}_\tau(\chi), T^\tau_\tau(\chi)\right)$ is just the identity $\mathrm{id} \in \mathrm{End}(M(\tau))$.

By means of the uncanonical definition of the triangle function we deduce

$$\begin{aligned}\Delta(-\nu,\nu;w_0)(\tau) &= \mathrm{Nat}(-\nu,\nu;w_0)(\tau)\circ\big(\mathrm{Nat}(-\nu,\nu;e)(\tau)\big)^{-1}(\mathrm{id}_{M(\tau)})\\ &= \big(\mathrm{Nat}(-\nu,\nu;w_0)(\tau)\big)(\mathrm{adj}^2)\end{aligned}$$

and the Theorem of Bernstein-Gelfand implies that the right hand side of the following diagram commutes:

$$(*)\qquad\begin{tikzcd} M(w_0\cdot\tau) \arrow[r, "\mathrm{adj}^1_{M(w_0\cdot\tau)}"] \arrow[dr, "\mathrm{id}"'] & T^{\tau}_{\tau+\nu}T^{\tau+\nu}_{\tau}M(w_0\cdot\tau) \arrow[r, "\mathrm{adj}^2_{M(w_0\cdot\tau)}"] \arrow[d, "\phi_{w_0}(w_0\cdot\tau)"] & M(w_0\cdot\tau) \\ & M(w_0\cdot\tau) \arrow[ur, "\Delta(-\nu,\nu;w_0)(\tau)"'] & \end{tikzcd}$$

The commutativity of the left hand side can be shown in an analogous way. We thus obtain for dominant ν

$$\mathrm{adj}^2_{M(w_0\cdot\tau)}\circ\mathrm{adj}^1_{M(w_0\cdot\tau)} = \Delta(-\nu,\nu;w_0)(\tau)$$

and together with the equation $\mathrm{adj}^2_{M(w_0\cdot\tau)}\circ\mathrm{adj}^1_{M(w_0\cdot\tau)} = \mathrm{tr}^{M(w_0\cdot\tau)}_{E(\nu)}(\mathrm{pr}_{\chi(\tau+\nu)})$ from section 5.3.2 we deduce Theorem 2. □

5.4. **The case $M(\tau)\longrightarrow T^{\tau}_{\tau+\nu}T^{\tau+\nu}_{\tau}M(\tau)\longrightarrow M(\tau)$.**

Corollary of the proof. *Let $\nu\in P^+$ be dominant and $\tau\in\mathfrak{h}^*$ generic. Then the composition*

$$M(\tau)\xrightarrow{\mathrm{adj}^1_{M(\tau)}} T^{\tau}_{\tau+\nu}T^{\tau+\nu}_{\tau}M(\tau)\xrightarrow{\mathrm{adj}^2_{M(\tau)}} M(\tau)$$

is just multiplication with $s(\tau) := \prod_{\alpha\in R^+}\frac{\langle\tau+\nu+\rho,\alpha^\vee\rangle}{\langle\tau+\rho,\alpha^\vee\rangle}$, where s is considered as an element in $\mathrm{Quot}(S(\mathfrak{h}))$*.*

Proof. The commutativity of the diagram $(*)$ in the proof of Theorem 2 is equivalent to the commutativity of

$$\begin{tikzcd} M(\tau) \arrow[r, "\mathrm{adj}^1_{M(\tau)}"] \arrow[dr, "\Delta(-\nu,\nu;w_0)(\tau)"'] & T^{\tau}_{\tau+\nu}T^{\tau+\nu}_{\tau}M(\tau) \arrow[r, "\mathrm{adj}^2_{M(\tau)}"] \arrow[d, "\phi_e(\tau)"] & M(\tau) \\ & M(\tau) \arrow[ur, "\mathrm{id}"'] & \end{tikzcd}$$

Hence $(\mathrm{adj}^2\circ\mathrm{adj}^1)_{M(\tau)} = \Delta(-\nu,\nu;w_0)(\tau) = \prod_{\alpha\in R^+}\frac{\langle\tau+\nu+\rho,\alpha^\vee\rangle}{\langle\tau+\rho,\alpha^\vee\rangle} = s(\tau)$. □

6. Calculation of Δ

Theorem 3. *For $\lambda\in\mathfrak{h}^*$ set $\bar\alpha(\lambda) := 1$ if $\langle\lambda,\alpha^\vee\rangle < 0$ and $\bar\alpha(\lambda) := 0$ if $\langle\lambda,\alpha^\vee\rangle\geq 0$. Let $\nu,\mu\in P$ be integral weights and $x\in\mathcal{W}$. Then there exists a constant $c\in k^\times$, independent of τ,ν,μ and x, such that*

$$\Delta(\mu,\nu;x)(\tau-\rho) = c\prod_{\substack{\alpha\in R^+\\ \text{with } x\alpha\in R^-}}\frac{\langle\tau,\alpha^\vee\rangle^{\bar\alpha(\nu)}\langle\tau+\nu,\alpha^\vee\rangle^{\bar\alpha(\mu)}\langle\tau+\nu+\mu,\alpha^\vee\rangle^{\bar\alpha(\nu+\mu)}}{\langle\tau,\alpha^\vee\rangle^{\bar\alpha(\nu+\mu)}\langle\tau+\nu,\alpha^\vee\rangle^{\bar\alpha(\nu)}\langle\tau+\nu+\mu,\alpha^\vee\rangle^{\bar\alpha(\mu)}}.$$

Proof. Postponed. □

For an integral weight $\nu \in P$ define the map $\delta_\nu \in \mathcal{P}(\mathfrak{h}^*)$ as in Section 3.2 by

$$\delta_\nu(\tau) := \prod_{\alpha \in R^+} \prod_{0 \le n_\alpha < -\langle \nu, \alpha^\vee \rangle} (\langle \tau + \rho, \alpha^\vee \rangle - n_\alpha) \ .$$

Lemma 11. *Let $\pi(\tau)$ be the product on the right hand side of the equation in Theorem 3. Then*

$$\pi(\tau + \rho) = \pm \frac{\delta_\nu(\tau)\delta_\mu(\tau+\nu)\delta_{x(\nu+\mu)}(x \cdot \tau)}{\delta_{x\nu}(x \cdot \tau)\delta_{x\mu}(x \cdot (\tau + \nu))\delta_{\nu+\mu}(\tau)}.$$

Proof. Let $x \in \mathcal{W}$ be fixed and let $\alpha \in R^+$ be a positive root. We then have in $\delta_\nu(\tau)$ the factor $\prod_{0 \le n < -\langle \nu, \alpha^\vee \rangle}(\langle \tau + \rho, \alpha^\vee \rangle - n)$. Suppose now that $x\alpha$ is also a positive root. Then we obtain in $\delta_{x\nu}(x \cdot \tau)$ the factor

$$\begin{aligned}\prod_{0 \le n < -\langle x\nu, x\alpha^\vee \rangle} (\langle x \cdot \tau + \rho, x\alpha^\vee \rangle - n) &= \prod_{0 \le n < -\langle \nu, \alpha^\vee \rangle} (\langle x(\tau + \rho), x\alpha^\vee \rangle - n) \\ &= \prod_{0 \le n < -\langle \nu, \alpha^\vee \rangle} (\langle \tau + \rho, \alpha^\vee \rangle - n)\end{aligned}$$

and hence in the quotient $\delta_\nu(\tau)/\delta_{x\nu}(x \cdot \tau)$ all the products over $\alpha \in R^+$ with $x\alpha \in R^+$ cancel out. Let now $\alpha \in R^+$ such that $x\alpha \notin R^+$, i.e. $-x\alpha$ is a positive root. In this case we have in $\delta_{x\nu}(x \cdot \tau)$ the factor

$$\begin{aligned}\prod_{0 \le n < -\langle x\nu, -x\alpha^\vee \rangle} (\langle x \cdot \tau + \rho, -x\alpha^\vee \rangle - n) &= \prod_{0 \le n < \langle \nu, \alpha^\vee \rangle} (-\langle \tau + \rho, \alpha^\vee \rangle - n) \\ &= (-1)^{\langle \nu, \alpha^\vee \rangle} \prod_{0 \le n < \langle \nu, \alpha^\vee \rangle} (\langle \tau + \rho, \alpha^\vee \rangle + n)\end{aligned}$$

and taken together we get

$$\frac{\delta_\nu(\tau)}{\delta_{x\nu}(x \cdot \tau)} = \prod_{\substack{\alpha \in R^+ \\ \text{with } x\alpha \in R^-}} (-1)^{\langle \nu, \alpha^\vee \rangle} \frac{\prod_{0 \le n_\alpha < -\langle \nu, \alpha^\vee \rangle}(\langle \tau + \rho, \alpha^\vee \rangle - n_\alpha)}{\prod_{0 \le n_\alpha < \langle \nu, \alpha^\vee \rangle}(\langle \tau + \rho, \alpha^\vee \rangle + n_\alpha)}.$$

Considering then the different cases according to whether $\langle \nu, \alpha^\vee \rangle, \langle \mu, \alpha^\vee \rangle$ and $\langle \nu + \mu, \alpha^\vee \rangle$ are positive or negative, we obtain the closed formula

$$\frac{\delta_\nu(\tau)\delta_\mu(\tau+\nu)\delta_{x(\nu+\mu)}(x \cdot \tau)}{\delta_{x\nu}(x \cdot \tau)\delta_{x\mu}(x \cdot (\tau + \nu))\delta_{\nu+\mu}(\tau)} = \varepsilon \ \pi(\tau + \rho),$$

where $\varepsilon := \prod_{\substack{\alpha \in R^+ \\ \text{mit } x\alpha \in R^-}} (-1)^{(\bar{\alpha}(\nu)\langle \nu, \alpha^\vee \rangle + \bar{\alpha}(\mu)\langle \mu, \alpha^\vee \rangle - \bar{\alpha}(\nu+\mu)\langle \mu+\nu, \alpha^\vee \rangle)} = \pm 1.$ ∎

Now, let $\tau \in \mathfrak{h}^*$ be a generic weight and recall (see 4.3) the map $\mathrm{Nat}_x(\tau) := \mathrm{Nat}(\mu, \nu; x)(\tau)$:

$$\begin{aligned}&\mathrm{Hom}_{\mathcal{M}(\chi(\tau)) \to} \left(T_{\tau+\nu}^{\tau+\nu+\mu} \circ T_\tau^{\tau+\nu}, T_\tau^{\tau+\nu+\mu}\right) \\ &\qquad \xrightarrow{\sim} \mathrm{Hom}_{\mathfrak{g}} \left(T_{\tau+\nu}^{\tau+\nu+\mu} T_\tau^{\tau+\nu} M(x \cdot \tau), T_\tau^{\tau+\nu+\mu} M(x \cdot \tau)\right) \\ &\qquad \xrightarrow{\sim} \mathrm{Hom}_{\mathfrak{g}} \left(M(x \cdot (\tau + \nu + \mu)), M(x \cdot (\tau + \nu + \mu))\right) \\ &\qquad \xrightarrow{\sim} k\end{aligned}$$

Denote the pre-image of $1 \in k$ under this isomorphism by the natural transformation $G^x(\tau) := (\mathrm{Nat}_x(\tau))^{-1}(1) \in \mathrm{Hom}_{\mathcal{M}(\chi(\tau))\to}(T_{\tau+\nu}^{\tau+\nu+\mu} \circ T_{\tau}^{\tau+\nu}, T_{\tau}^{\tau+\nu+\mu})$.

Lemma 12. *For generic $\tau \in \mathfrak{h}^*$ we have $G^e(\tau) = \Delta(\mu,\nu;x)(\tau)G^x(\tau)$.*

Proof. By definition, $\Delta(\mu,\nu;x)(\tau) = \mathrm{Nat}_x(\tau) \circ (\mathrm{Nat}_e(\tau))^{-1}(1)$ and therefore

$$\begin{aligned} G^e(\tau) &= (\mathrm{Nat}_x(\tau))^{-1} \circ \mathrm{Nat}_x(\tau) \circ (\mathrm{Nat}_e(\tau))^{-1}(1) \\ &= (\mathrm{Nat}_x(\tau))^{-1}(\Delta(\mu,\nu;x)(\tau)) \\ &= \Delta(\mu,\nu;x)(\tau)(\mathrm{Nat}_x(\tau))^{-1}(1) \\ &= \Delta(\mu,\nu;x)G^x(\tau). \end{aligned}$$

□

Let now integral weights $\nu_1,\dots,\nu_n$ be given such that $\sum_{i=1}^n \nu_i = 0$. We set the translation functor $T(\nu_i) := T_{\tau+\nu_1+\cdots+\nu_{i-1}}^{\tau+\nu_1+\cdots+\nu_i}$ when there is no ambiguity of the respective categories. For $\lambda \in P$ an integral weight and τ generic there are isomorphisms (see 4.3) $F_{x\lambda}(x\cdot\tau) : M(x\cdot(\tau+\lambda)) \xrightarrow{\sim} T_\tau^{\tau+\lambda}M(x\cdot\tau)$, such that $(F_{x\lambda}(x\cdot\tau))(v_{x\cdot\tau}) = \mathrm{pr}_{\chi(\tau+\lambda)}(\dot{x}e_\lambda \otimes v_{x\cdot\tau})$. Here, $e_\lambda \in E(\lambda)_\lambda$ is a fixed chosen extremal weight vector. Since $\sum_{i=1}^n \nu_i = 0$, we can compose these isomorphisms to obtain an isomorphism $M(x\cdot\tau) \cong T(\nu_n)\cdots T(\nu_1)M(x\cdot\tau)$ and thus also

$$\begin{aligned} \mathrm{Hom}_{\mathfrak{g}}\big(M(x\cdot\tau), M(x\cdot\tau)\big) &\xrightarrow{\sim} \mathrm{Hom}_{\mathfrak{g}}\big(M(x\cdot\tau), T(\nu_n)\cdots T(\nu_1)M(x\cdot\tau)\big) \\ &\hookrightarrow \mathrm{Hom}_k\big(\mathfrak{U}(\mathfrak{n}^-), E(\nu_n)\otimes\cdots\otimes E(\nu_1)\otimes\mathfrak{U}(\mathfrak{n}^-)\big), \end{aligned}$$

where we have identified the Verma module $M(x\cdot\tau)$ with $\mathfrak{U}(\mathfrak{n}^-)$. Call now the image of the identity on $M(x\cdot\tau)$ under this map $h^x(\nu_1,\dots,\nu_n)(\tau)$ and let $\mathcal{U} \subset \mathfrak{h}^*$ be the set of all generic weights. We then obtain for all $x \in \mathcal{W}$ a function

$$\begin{array}{rccc} h^x(\nu_1,\dots,\nu_n): & \mathcal{U} & \to & \mathrm{Hom}_k\big(\mathfrak{U}(\mathfrak{n}^-), E(\nu_n)\otimes\cdots\otimes E(\nu_1)\otimes\mathfrak{U}(\mathfrak{n}^-)\big) \\ & \tau & \mapsto & h^x(\nu_1,\dots,\nu_n)(\tau), \end{array}$$

such that $h^x(\nu_1,\dots,\nu_n)(\tau)$ maps the element $v_{x\cdot\tau} \in M(x\cdot\tau) \cong \mathfrak{U}(\mathfrak{n}^-)$ to the element $\mathrm{pr}_{\chi(\tau+\nu_1+\cdots+\nu_n)}(\dot{x}e_{\nu_n}\otimes(\cdots\otimes\mathrm{pr}_{\chi(\tau+\nu_1)}(\dot{x}e_{\nu_1}\otimes v_{x\cdot\tau}))\cdots)$ for fixed vectors $e_{\nu_i} \in E(\nu_i)_{\nu_i}$.

Lemma 13. *Set $d^x(\nu_1,\dots,\nu_n)(\tau) := \delta_{x\nu_1}(x\cdot\tau)\delta_{x\nu_2}(x\cdot(\tau+\nu_1))\cdots\delta_{x\nu_n}(x\cdot(\tau+\nu_1+\cdots+\nu_{n-1}))$. Then the map $d^x(\nu_1,\dots,\nu_n)h^x(\nu_1,\dots,\nu_n)$ is algebraic on $\mathcal{U}$ and there exists an algebraic extension on $\mathfrak{h}^*$ whose set of zeros has codimension ≥ 2.*

Remark. Here, we call a map $a : \mathcal{U} \to W$ to a vector space W *algebraic*, if it is a morphism of varieties. Of course, this is defined only if $\dim W < \infty$. In our case however, the image of $h^x(\nu_1,\dots,\nu_n)$ is always contained in a finite dimensional subspace of $\mathrm{Hom}_k\big(\mathfrak{U}(\mathfrak{n}^-), E(\nu)\otimes\cdots\otimes E(\nu_1)\otimes\mathfrak{U}(\mathfrak{n}^-)\big)$ and we may thus regard $h^x(\nu_1,\dots,\nu_n)$ as a map between varieties.

Proof. Let $\nu \in P$ be an integral weight and recall the maps

$$\begin{array}{rccc} f_\nu: & \mathfrak{h}^* & \longrightarrow & E(\nu)\otimes M(\tau) \cong E(\nu)\otimes\mathfrak{U}(\mathfrak{n}^-) \\ & \tau & \mapsto & \mathrm{pr}_{\chi(\tau+\nu)}(e_\nu\otimes v_\tau) \end{array}$$

Let now $x \in \mathcal{W}$ be fixed. For generic τ define then the map $a^x_\nu(\tau)$ by

$$\begin{array}{rccc} a^x_\nu(\tau): & M(x\cdot\tau) \cong \mathfrak{U}(\mathfrak{n}^-) & \longrightarrow & E(\nu)\otimes M(x\cdot\tau) \cong E(\nu)\otimes\mathfrak{U}(\mathfrak{n}^-) \\ & v_{x\cdot\tau} & \mapsto & \delta_{x\nu}(x\cdot\tau) f_{x\nu}(x\cdot\tau) \end{array}$$

where $f_{x\nu}(x\cdot\tau) = \operatorname{pr}_{\chi(\tau+\nu)}(\dot{x}e_\nu \otimes v_{x\cdot\tau})$. Since f_ν is algebraic on $\mathcal{U}$ (see Theorem 1), we obtain in this way also an algebraic map

$$\begin{array}{rccc} a^x_\nu: & \mathcal{U} & \longrightarrow & \operatorname{Hom}_k\big(\mathfrak{U}(\mathfrak{n}^-), E(\nu)\otimes\mathfrak{U}(\mathfrak{n}^-)\big) \\ & \tau & \mapsto & a^x_\nu(\tau) \end{array}$$

Here again, the image of a^x_ν is contained in a finite dimensional subspace of $\operatorname{Hom}_k\big(\mathfrak{U}(\mathfrak{n}^-), E(\nu)\otimes\mathfrak{U}(\mathfrak{n}^-)\big)$ and we regard a^x_ν in this way as a map between varieties.

According to Theorem 1 there is an algebraic extension of $\delta_\nu f_\nu$ on $\mathfrak{h}^*$, which vanishes only on a set of codimension ≥ 2. Therefore, also a^x_ν has such an algebraic extension and we call it again a^x_ν. For generic τ we then have $(a^x_\nu(\tau))(v_{x\cdot\tau}) = \delta_{x\nu}(x\cdot\tau)\operatorname{pr}_{\chi(\tau+\nu)}(\dot{x}e_\nu\otimes v_{x\cdot\tau})$. Note that for all $\tau\in\mathfrak{h}^*$ the vector $(a^x_\nu(\tau))(v_{x\cdot\tau}) \in E(\nu)\otimes M(x\cdot\tau)$ generates a Verma module with highest weight $x\cdot(\tau+\nu)$. The image of $a^x_\nu(\tau)$ is thus always contained in a Verma module $M(x\cdot(\tau+\nu)) \subset E(\nu)\otimes M(x\cdot\tau)$ and we can identify the element $(a^x_\nu(\tau))(v_{x\cdot\tau})$ with the canonical generator $v_{x\cdot(\tau+\nu)} \in M(x\cdot(\tau+\nu))$. Using the isomorphism $M(x\cdot(\tau+\nu)) \cong \mathfrak{U}(\mathfrak{n}^-)$, we may then apply the map $a^x_{\nu'}(\tau+\nu)$ for another weight $\nu'\in P$. In particular, we can concatenate the maps $a^x_{\nu_n}(\tau+\nu_1+\cdots+\nu_{n-1})\circ\cdots\circ a^x_{\nu_2}(\tau+\nu_1)\circ a^x_{\nu_1}(\tau)$ and obtain in this way an algebraic map

$$\begin{array}{rccc} a: & \mathfrak{h}^* & \longrightarrow & \operatorname{Hom}_k\big(\mathfrak{U}(\mathfrak{n}^-), E(\nu_n)\otimes\cdots E(\nu_1)\otimes\mathfrak{U}(\mathfrak{n}^-)\big) \\ & \tau & \mapsto & a^x_{\nu_n}(\tau+\nu_1+\cdots+\nu_{n-1})\circ\cdots\circ a^x_{\nu_2}(\tau+\nu_1)\circ a^x_{\nu_1}(\tau) \end{array}$$

which vanishes only on a set of codimension ≥ 2 and which maps a generic weight τ to $a(\tau) = \delta_{x\nu_1}(x\cdot\tau)\delta_{x\nu_2}(x\cdot(\tau+\nu_1))\cdots\delta_{x\nu_n}(x\cdot(\tau+\nu_1+\cdots\nu_{n-1}))$ $h^x(\nu_1,\dots,\nu_n)(\tau) = d^x(\nu_1,\dots,\nu_n)(\tau)h^x(\nu_1,\dots,\nu_n)(\tau)$. We thus obtain the map a as the desired algebraic extension. □

According to the Theorem of Bernstein-Gelfand we may interpret for generic τ the maps $h^x(\nu_1,\dots,\nu_n)(\tau) \in \operatorname{Hom}_{\mathfrak{g}}\big(M(x\cdot\tau), T(\nu_n)\cdots T(\nu_1)M(x\cdot\tau)\big)$ $\cong \operatorname{Hom}_{\mathcal{M}(\chi(\tau))\to}\big(\operatorname{Id}, T(\nu_n)\cdots T(\nu_1)\big)$ as natural transformations of functors. Set now

$$\begin{array}{ll} h_1 := h^x(\nu+\mu,-\mu,-\nu)(\tau) & \in \operatorname{Hom}_{\mathcal{M}(\chi(\tau))\to}\big(\operatorname{Id}, T(-\nu)T(-\mu)T(\nu+\mu)\big), \\ h_2 := h^x(-\nu,\nu)(\tau+\nu) & \in \operatorname{Hom}_{\mathcal{M}(\chi(\tau+\nu))\to}\big(\operatorname{Id}, T(\nu)T(-\nu)\big), \\ h_3 := h^x(-\mu,\mu)(\tau+\nu+\mu) & \in \operatorname{Hom}_{\mathcal{M}(\chi(\tau+\nu+\mu))\to}\big(\operatorname{Id}, T(\mu)T(-\mu)\big) \end{array}$$

and consider the natural transformations

$$\begin{tikzcd} T(\mu)T(\nu) \arrow[r, "G^x(\tau)"] \arrow[d, "T(\mu)T(\nu)h_1"'] & T(\nu+\mu) \\ T(\mu)T(\nu)T(-\nu)T(-\mu)T(\nu+\mu) \arrow[r, "(T(\mu)h_2)^{-1}"] & T(\mu)T(-\mu)T(\nu+\mu) \arrow[u, "(h_3)^{-1}"'] \end{tikzcd}$$

The diagram commutes, since $(h_3)^{-1}\circ(T(\mu)h_2)^{-1}\circ T(\mu)T(\nu)h_1$ as well as $G^x(\tau)$ imply the identity on $M(x\cdot(\tau+\nu+\mu))$ under the isomorphism

$$\begin{aligned}
&\mathrm{Hom}_{\mathcal{M}(\chi(\tau))\to}\big(T(\mu)\circ T(\nu), T(\nu+\mu)\big)\\
&\qquad\qquad\xrightarrow{\sim}\mathrm{Hom}_{\mathfrak{g}}\big(T(\mu)T(\nu)M(x\cdot\tau), T(\nu+\mu)M(x\cdot\tau)\big)\\
&\qquad\qquad\xrightarrow{\sim}\mathrm{Hom}_{\mathfrak{g}}\big(M(x\cdot(\tau+\nu+\mu)), M(x\cdot(\tau+\nu+\mu))\big).
\end{aligned}$$

Lemma 14. *Set $D^x(\tau) := \delta_{x(\nu+\mu)}(x\cdot\tau)/\delta_{x\nu}(x\cdot\tau)\delta_{x\mu}(x\cdot(\tau+\nu))$. Then the map D^xG^x is algebraic on $\mathcal{U}$ and there exists an algebraic extension on $\mathfrak{h}^*$ whose set of zeros has codimension ≥ 2.*

Proof. Since the maps $d^x(\nu_1,\dots,\nu_n)h^x(\nu_1,\dots,\nu_n)$ have such algebraic extensions (Lemma 13), the commutativity of the above diagram implies that also D^xG^x has such an algebraic extension, where

$$\begin{aligned}
D^x(\tau) &= d^x(\nu+\mu,-\mu,-\nu)(\tau)d^x(-\nu,\nu)(\tau+\nu)d^x(-\mu,\mu)(\tau+\nu+\mu)\\
&= \delta_{x(\nu+\mu)}(x\cdot\tau)/\delta_{x\nu}(x\cdot\tau)\delta_{x\mu}(x\cdot(\tau+\nu)).
\end{aligned}$$

□

Finally, we come to the

Proof of Theorem 3. By Lemma 11 it suffices to show that $\Delta(\mu,\nu;x)(\tau) = c\ \pi(\tau+\rho)$ for a non-vanishing constant c. Note, that $(D^x/D^e)(\tau) = \pm\pi(\tau+\rho)$. We deduce from Lemma 12 that for generic weights $D^xD^eG^e = \Delta(\mu,\nu;x)D^eD^xG^x$. Since D^eG^e as well as D^xG^x have algebraic extensions on $\mathfrak{h}^*$ which vanish only on a set of codimension ≥ 2 (Lemma 14), it follows that there is a constant $c \in k^\times$, independent of τ,μ,ν and x, such that

$$c\,\Delta(\mu,\nu;x)(\tau) = (D^x/D^e)(\tau) = \pm\pi(\tau+\rho).$$

□

7. **Outlook**

7.1. **Identities.** There are many nice identities for the triangle functions. Obvious are the

Normalization identities.

$$\Delta(\mu,\nu;e) = 1 \qquad\qquad \Delta(0,\nu;x) = 1 \qquad\qquad \Delta(\nu,0;x) = 1$$

By means of Theorem 3 or directly with the definition of Δ one then checks for $\nu,\mu,\eta \in P$ and $x, y \in \mathcal{W}$:

Decomposition identity. $\Delta(\eta+\mu,\nu;x)(\tau)\ \Delta(\eta,\mu;x)(\tau+\nu) = \Delta(\eta,\mu+\nu;x)(\tau)\ \Delta(\mu,\nu;x)(\tau)$

Rotation identity. $\Delta(y\mu,y\nu;x)(y\cdot\tau) = (\Delta(\mu,\nu;y)(\tau))^{-1}\ \Delta(\mu,\nu;xy)(\tau)$

Flat triangle identity. *Let $\nu,\mu \in P$ be in the closure of a Weyl chamber. Then $\Delta(\mu,\nu;x) = 1$ for all $x \in \mathcal{W}$.*

7.2. **Generalizations.**

7.2.1. *The Weyl group parameter.* Let $\nu, \mu \in P$ and $x, y \in \mathcal{W}$. Instead of applying the translation functors to the Verma modules $M(\tau)$ and $M(x \cdot \tau)$, we may choose the Verma modules $M(x \cdot \tau)$ and $M(y \cdot \tau)$. We then define a generalized triangle function Δ_{g} by

$$\Delta_{\mathrm{g}}(\mu, \nu; y, x)(\tau) := \det\left(yx^{-1} \circ \mathrm{nat}(\mu, \nu; x)(\tau) \circ (\mathrm{nat}(\mu, \nu; y)(\tau))^{-1}\right).$$

We now have $\Delta_{\mathrm{g}}(\mu, \nu; e, x)(\tau) = \Delta(\mu, \nu; x)(\tau)$ and going back to the definition of Δ_{g} we deduce the identities

(I1) $\Delta_{\mathrm{g}}(\mu, \nu; y, x)(\tau) = (\Delta(\mu, \nu; y)(\tau))^{-1}\ \Delta(\mu, \nu; x)(\tau)$
(I2) $\Delta_{\mathrm{g}}(\mu, \nu; y, x)(\tau) = (\Delta_{\mathrm{g}}(\mu, \nu; x, y)(\tau))^{-1}$

and

(I3) $\Delta_{\mathrm{g}}(\mu, \nu; y, x)\ \Delta_{\mathrm{g}}(\mu, \nu; x, z) = \Delta_{\mathrm{g}}(\mu, \nu; y, z)$.

The equivalent statement to the Rotation identity is obtained by comparing (I1) with

(I4) $\Delta_{\mathrm{g}}(\mu, \nu; y, x)(\tau) = \Delta(y\mu, y\nu; xy^{-1})(y \cdot \tau)$.

7.2.2. *Number of translations.* The triangle functions measure in a subtle way the relation between the two translation functors $T_{\tau+\nu}^{\tau+\nu+\mu} \circ T_{\tau}^{\tau+\nu}$ and $T_{\tau}^{\tau+\nu+\mu}$. Therefore the triangle

$$\begin{tikzcd}
& \bullet \arrow[dr, "\mu"] & \\
\tau \arrow[ur, "\nu"] \arrow[rr, "\nu+\mu"'] & & \bullet
\end{tikzcd}$$

Let now integral weights $\nu_1, \dots, \nu_n$ and $\mu_1, \dots, \mu_m \in P$ be given such that $\sum_{i=1}^{n} \nu_i = \sum_{j=1}^{m} \mu_j =: p$. Call then $T(\nu_i)$ the translation functor

$$\begin{array}{rccc}
T(\nu_i): & \mathcal{M}^\infty\big(\chi(\tau + \nu_1 + \cdots + \nu_{i-1})\big) & \longrightarrow & \mathcal{M}^\infty\big(\chi(\tau + \nu_1 + \cdots + \nu_i)\big) \\
& M & \mapsto & \mathrm{pr}_{\chi(\tau+\nu_1+\cdots+\nu_i)}(E(\nu_i) \otimes M)
\end{array}$$

and define similarly the translation functor $T(\mu_i)$. We now want to compare the functors $T(\nu_n) \circ \cdots \circ T(\nu_1)$ and $T(\mu_m) \circ \cdots \circ T(\mu_1)$ with each other and instead of a triangle of translations we thus have now the following situation:

$$\begin{tikzcd}
& \bullet \arrow[r] & \bullet \arrow[r, dotted, no head] & \bullet \arrow[dr, "\mu_m"] & \\
\tau \arrow[ur, "\mu_1"] \arrow[dr, "\nu_1"'] & & & & \tau + p \\
& \bullet \arrow[r] & \bullet \arrow[r, dotted, no head] & \bullet \arrow[ur, "\nu_n"'] &
\end{tikzcd}$$

We start by defining a map $\bar{\mathrm{nat}}(\mu_m, \dots, \mu_1; \nu_n, \dots, \nu_1; x)(\tau)$ for $x \in \mathcal{W}$ and generic weight τ analogously to the definition of $\mathrm{nat}(\mu, \nu; x)(\tau)$ (see page 14) as the composition

$$
\begin{aligned}
&\mathrm{Hom}_{\mathcal{M}(\chi(\tau))\rightarrow}\big(T(\mu_m)\circ\cdots\circ T(\mu_1), T(\nu_n)\circ\cdots\circ T(\nu_1)\big)\\
&\xrightarrow{\sim}\mathrm{Hom}_{\mathfrak{g}}\big(T(\mu_m)\cdots T(\mu_1)M(x\cdot\tau), T(\nu_n)\cdots T(\nu_1)M(x\cdot\tau)\big)\\
&\xrightarrow{\sim}\mathrm{Hom}_{\mathfrak{g}}\big(E_{\mu_m}\hat{\otimes}\cdots E_{\mu_1}\hat{\otimes}M(x\cdot(\tau+p)), E_{\nu_n}\hat{\otimes}\cdots E_{\nu_1}\hat{\otimes}M(x\cdot(\tau+p))\big)\\
&\xrightarrow{\sim}E^*_{\mu_m}\otimes\cdots\otimes E^*_{\mu_1}\otimes E_{\nu_n}\otimes\cdots\otimes E_{\nu_1}
\end{aligned}
$$

Here, we wrote E_μ for $E(\mu)_{x\mu}$. We then obtain a generalized triangle function $\diamond$ by

$$
\begin{aligned}
\diamond(\mu_m,\dots\mu_1;\nu_n,\dots\nu_1;x)(\tau) := \det\big(&x^{-1}\circ \bar{\mathrm{nat}}(\mu_m,\dots\mu_1;\nu_n,\dots\nu_1;x)(\tau)\\
&\circ(\bar{\mathrm{nat}}(\mu_m,\dots\mu_1;\nu_n,\dots\nu_1;e)(\tau))^{-1}\big).
\end{aligned}
$$

Obviously we have $\diamond(\mu,\nu;\nu+\mu;x)=\Delta(\mu,\nu;x)$ as well as

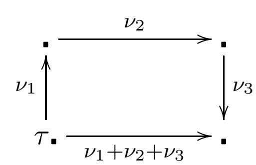
$$\diamond(\mu_m,\dots,\mu_1;\nu_n,\dots,\nu_1;x)(\tau)=(\diamond(\nu_n,\dots,\nu_1;\mu_m,\dots,\mu_1;x)(\tau))^{-1}$$

and we can reduce the calculation of $\diamond$ to the calculation of Δ by means of the

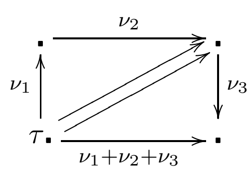
Split identities.

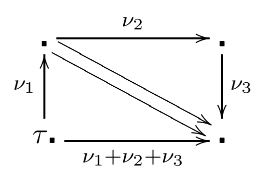
$$
\begin{aligned}
(1)\quad \diamond(\nu_3,\nu_2,\nu_1;\nu_3+\nu_2+\nu_1;x)(\tau) &= \Delta(\nu_2,\nu_1;x)(\tau)\Delta(\nu_3,\nu_1+\nu_2;x)(\tau)\\
&=\Delta(\nu_2+\nu_3,\nu_1;x)(\tau)\Delta(\nu_3,\nu_2;x)(\tau+\nu_1)
\end{aligned}
$$

$$(2)\qquad \diamond(\mu_2,\mu_1;\nu_2,\nu_1;x)(\tau)=\Delta(\mu_2,\mu_1;x)(\tau)\big(\Delta(\nu_2,\nu_1;x)(\tau)\big)^{-1}$$

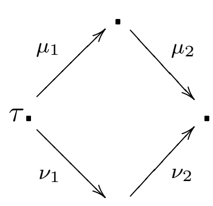
The first identity just means that we can split 

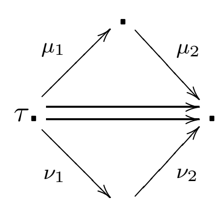
into

triangles according to 

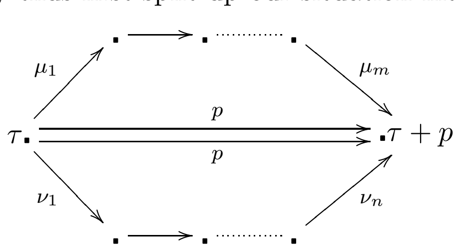
or 

.

This is just the Decomposition identity in 7.1. The second equation describes the decomposition of 

in 

Inductively, we may thus first split up our situation into



In formulae :

$$\begin{aligned}\diamond(\mu_m,\dots\mu_1;\nu_n,\dots\nu_1;x)(\tau) &= \diamond(\mu_m,\dots\mu_1;p;x)(\tau)\,\diamond(p;\nu_n,\dots\nu_1;x)(\tau)\\ &= \diamond(\mu_m,\dots\mu_1;p;x)(\tau)\big(\diamond(\nu_n,\dots\nu_1;p;x)(\tau)\big)^{-1}\end{aligned}$$

and in order to calculate the $\diamond$-functions it suffices thus to know them in the special case $\diamond(\mu_m,\dots,\mu_1;\sum_{j=1}^m\mu_j;x)$. This situation can then be reduced to triangles by decomposing it into

$$\begin{tikzcd}
 & \cdot \arrow[r] & \cdot \arrow[r, dotted, no head] & \cdot \arrow[r, "\mu_{m-1}"] & \cdot \arrow[dr, "\mu_m"] & \\
\tau \arrow[ur, "\mu_1"] \arrow[urrrr, shift left] \arrow[urrrr, shift right] \arrow[rrrrr, "\mu_1+\cdots+\mu_m"] & & & & & \tau+p
\end{tikzcd}$$

or into

$$\begin{tikzcd}
 & \cdot \arrow[r] \arrow[drrrr, shift left] \arrow[drrrr, shift right] & \cdot \arrow[r, dotted, no head] & \cdot \arrow[r, "\mu_{m-1}"] & \cdot \arrow[dr, "\mu_m"] & \\
\tau \arrow[ur, "\mu_1"] \arrow[rrrrr, "\mu_1+\cdots+\mu_m"] & & & & & \tau+p
\end{tikzcd}$$

Inductively one can then prove

Proposition. *Let $\mu_1,\dots,\mu_m\in P$ be integral weights and $x\in\mathcal{W}$. Then*

$$\begin{aligned}\diamond(\mu_m,\dots,\mu_1;\textstyle\sum_{j=1}^m\mu_j;x)(\tau) &= \prod_{k=2}^{m}\Delta(\mu_k,\textstyle\sum_{i=1}^{k-1}\mu_i;x)(\tau)\\ &= \prod_{k=1}^{m-1}\Delta(\textstyle\sum_{j=k+1}^{m}\mu_j,\mu_k;x)(\tau+\sum_{j=1}^{k-1}\mu_j)\end{aligned}$$

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